

STRING OPERATIONS ON RATIONAL GORENSTEIN SPACES

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ABSTRACT. Félix and Thomas developed string topology of Chas and Sullivan on simply-connected Gorenstein spaces. In this paper, we prove that the degree shifted homology of the free loop space of a simply-connected \mathbb{Q} -Gorenstein space with rational coefficient is a non-unital and non-counital Frobenius algebra by solving the up to constant problem. We also investigate triviality or non-triviality of the loop product and coproduct of particular Gorenstein spaces.

1. INTRODUCTION

Chas and Sullivan [1] introduced a new algebraic structure which is an intersection type product (called the loop product) on the shifted singular homology $\mathbb{H}_*(LM) = H_{*+d}(LM)$ of the free loop space $LM = \text{Map}(S^1, M)$ of any closed oriented d -manifold M . Cohen and Godin [4] generalized the product which is called string operations and showed that $H_*(LM)$ has the structure of a noncounital commutative Frobenius algebra. It thus gives rise to a 2-dimensional topological quantum field theory without counit. The coproduct of $H_*(LM)$ is called the loop coproduct.

After the appearance of [1], several authors have extended the theory of string topology. For instance, Chataur and Menichi [2] considered string topology of classifying spaces. They also proved in the article that the singular cohomology of the free loop space of the classifying space of a compact connected Lie group or a finite group with field coefficients is a homological conformal field theory. In [9], Félix and Thomas developed string topology on Gorenstein spaces. A Gorenstein space was defined by Félix, Halperin and Thomas in [6] and, for instance, closed oriented manifolds, the classifying spaces of connected Lie groups and Borel constructions of Poincaré duality spaces are Gorenstein spaces. The following question is proposed by Félix and Thomas; see [9, p.423].

Question. *Do the string operations of the homology of the free loop space of a simply-connected Gorenstein space give rise to a 2-dimensional topological quantum field theory?*

To answer the question, it is necessary to consider the “up to constant problem”. That is to say, there is a problem which involves the strict associative of the loop product and so on. The aim of this paper is to give one approach for solving the problem in rational coefficient case.

Let Dlp and $Dlcp$ be the dual loop product and the dual loop coproduct; see §2 and §3 for the precise definitions. Denote by $(H^*(LM; \mathbb{Q}))^\vee \cong H_*(LM; \mathbb{Q})$ the dual vector space of $H^*(LM; \mathbb{Q})$ and by

$$\begin{aligned} Lp &= (Dlp)^\vee : H_*(LM; \mathbb{Q}) \otimes H_*(LM; \mathbb{Q}) \longrightarrow H_*(LM; \mathbb{Q}), \\ Lcop &= (Dlcp)^\vee : H_*(LM; \mathbb{Q}) \longrightarrow H_*(LM; \mathbb{Q}) \otimes H_*(LM; \mathbb{Q}) \end{aligned}$$

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the duals of Dlp and $Dlcp$, respectively. The following theorem is the main result in this article. Proposition 3.1, 4.1 and 5.1 show the following identities (1), (2) and (3).

Theorem 1.1. *Let M be a simply-connected \mathbb{Q} -Gorenstein space of formal dimension d . Then,*

- (1) $Lp(Lp \otimes 1) = (-1)^d Lp(1 \otimes Lp)$,
- (2) $(Lcop \otimes 1)Lcop = (-1)^d (1 \otimes Lcop)Lcop$,
- (3) $Lcop \circ Lp = (-1)^d (Lp \otimes 1)(1 \otimes Lcop) = (-1)^d (1 \otimes Lp)(Lcop \otimes 1)$.

Fundamental and important maps which appear in string topology are decomposable with appropriate pull-back diagrams via the Eilenberg-Moore isomorphism; see [9] [12]. We shall build the proof of our main result Theorem 1.1 by combining the idea with tools in rational homotopy theory. Actually, the first use of such idea due to Kuribayashi, Menichi and the author [12] gives an advantage in the study of string operations.

Tamanoi [16, Theorem A] proved that if M is an oriented closed smooth manifold, both $(Lcop \otimes 1)Lcop$ and $(1 \otimes Lcop)Lcop$ vanish. However $(Lcop \otimes 1)Lcop$ is not necessarily trivial for Gorenstein spaces. We will give an example of non-trivial case of $(Lcop \otimes 1)Lcop$ in Example 6.5. The equation (3) of Theorem 1.1 is called Frobenius compatibility and compare with Tamanoi's result [16, Theorem 2.2].

We now define the product m on $\mathbb{H}_*(LM; \mathbb{Q}) = H_{*+d}(LM; \mathbb{Q})$ by

$$m(a \otimes b) = (-1)^{d(|a|+d)} Lp(a \otimes b).$$

for $a, b \in \mathbb{H}_*(LM; \mathbb{Q})$; see [5, Proposition 4] and [16, §2] for the sign. By Theorem 1.1 (1) and [12, Lemma 11.8], we have the following.

Corollary 1.2. *The shifted homology $\mathbb{H}_*(LM; \mathbb{Q}) = H_{*+d}(LM; \mathbb{Q})$ endowed with m is an associative graded algebra.*

Another purpose of this paper is to investigate of triviality or non-triviality of the loop (co)product of a simply-connected Gorenstein space. In [16], Tamanoi proved that the loop coproduct of a connected closed oriented manifold is trivial if its Euler characteristic is zero. Félix and Thomas [9] showed that, in rational coefficient, the loop product of the classifying space of a compact connected Lie group is trivial and the dual loop coproduct is non-trivial. For rational Gorenstein spaces, the torsion functor description of [12] enables us to obtain the following two results about triviality or nontriviality of the operations.

Proposition 1.3. *Let M be a simply-connected \mathbb{Q} -Gorenstein space with a minimal Sullivan model $(\Delta V, d)$. Suppose that the rational homotopy group of M is finite dimensional, that is, V is finite dimensional.*

- (1) *If V is generated by odd degree elements, then the loop product is non-trivial and the loop coproduct is trivial.*
- (2) *If V is generated by even degree elements, then the loop product is trivial and the loop coproduct is non-trivial.*

We now remark that if V is generated by odd degree elements, M is a Poincaré duality space since $H^*(M; \mathbb{Q})$ is finite dimensional. Therefore, by [12], $\mathbb{H}_*(LM; \mathbb{Q})$ is unital and hence the loop product is not trivial.

Proposition 1.4. *Let M be a simply-connected \mathbb{Q} -Gorenstein space whose minimal Sullivan model $(\Delta V, d)$ is pure. Moreover, assume that both V^{odd} and V^{even} are not zero.*

- (1) *If the differential d is zero, then both the loop product and the loop coproduct on $H_*(LM; \mathbb{Q})$ are trivial.*

(2) If $\dim V^{odd} > \dim V^{even}$, then the loop coproduct is trivial.

See Definition 6.3 for pure Sullivan models. For example, even dimensional spheres, complex projective spaces and homogeneous spaces have pure minimal Sullivan models. As a computational example, we consider the loop (co)product on the Borel construction $ES^1 \times_{S^1} \mathbb{C}P^2$. That gives the first example for which both of the product and the coproduct are non trivial in higher degree; see Example 6.5 and compare with the result [16, Theorem B (2)] due to Tamanoi.

The organization of this paper is as follows. In Section 2, we recall a fundamental definitions and facts for string topology of Gorenstein spaces. In Sections 3 and 4, we give the definition of dual loop product and coproduct. A proof of an associativity of the operations is also stated in the section. In Section 5, we show that the loop product and the loop coproduct satisfy Frobenius compatibility. A proof of Proposition 1.3, 1.4 and a computational example are presented in Section 6.

2. PRELIMINARIES

We first fix notation and terminology and recall some results for string topology on Gorenstein spaces. For a topological space X , let LX be the free loop space of X , let X^I be the space consisting of all continuous maps from the closed unit interval $I = [0, 1]$ to the space X and let $ev_0 : LX \rightarrow X$ be the evaluation map at 0. Denote by $LX \times_X LX$ the pull-back

$$\begin{array}{ccc} LX \times_X LX & \xrightarrow{pr_1} & LX \\ pr_2 \downarrow & & \downarrow ev_0 \\ LX & \xrightarrow{ev_0} & X. \end{array}$$

For a fixed $s \in [0, 1]$, let $ev_{i,s} : LX \times_X LX \rightarrow X$ be the map which is defined as $ev_{i,s} = ev_s \circ pr_i$ for $i = 1, 2$. In particular, we put $ev_0 = ev_{i,0} : LX \times_X LX \rightarrow X$.

We next recall the definition of Gorenstein spaces. Let A be a differential graded algebra, M and N differential graded A -modules. We denote by $\text{Ext}_A(M, N)$ the differential Ext in the sense of Moore, that is, the homology of $\text{Hom}_A(P, M)$, where P is a A -semifree resolution of M ; see [6, Appendix].

Definition 2.1 ([6, §3]). A differential graded augmented algebra over a field \mathbb{K} , (A, d) , is called a *Gorenstein algebra* of formal dimension d if

$$\dim \text{Ext}_A^*(\mathbb{K}, A) = \begin{cases} 0 & (* \neq d) \\ 1 & (* = d). \end{cases}$$

A path-connected space M is called a \mathbb{K} -Gorenstein space of formal dimension d if the normalized singular cochain algebra $C^*(M)$ with coefficients in \mathbb{K} is a Gorenstein algebra of formal dimension d .

For example, any \mathbb{K} -Poincaré duality space is a \mathbb{K} -Gorenstein space. Given $F \rightarrow E \rightarrow B$ a fibration of simply connected spaces of finite type over a field \mathbb{K} . Then, B is a \mathbb{K} -Gorenstein space and F is a \mathbb{K} -Poincaré duality space if and only if E is a \mathbb{K} -Gorenstein space ([6, Theorem 4.3], [14, Theorem 1.2]). In particular, the classifying space of a compact connected Lie group G is a Gorenstein space with formal dimension $-\dim G$.

The following is a key theorem for string topology on Gorenstein spaces.

Theorem 2.2 ([9, Theorem 12]). *Let M be a simply-connected \mathbb{K} -Gorenstein space of formal dimension d whose cohomology with coefficients in \mathbb{K} is of finite type. Then*

$$\text{Ext}_{C^*(M^{\times n})}^*(C^*(M), C^*(M^{\times n})) \cong H^{*-(n-1)d}(M)$$

where $C^*(M)$ is considered a $C^*(M^{\times n})$ -module via the diagonal map $\Delta : M \rightarrow M^{\times n}$.

In this paper, we consider the case that a ground field \mathbb{K} is the field of rational numbers \mathbb{Q} . Now, recall the fundamental facts on rational homotopy theory. For details about rational homotopy theory, see [7] or [8] for example. A *minimal Sullivan model* for a simply-connected space X with finite type is a free commutative differential graded algebra over \mathbb{Q} , $(\Lambda V, d)$, with a graded vector space $V = \bigoplus_{i \geq 2} V^i$ over \mathbb{Q} where each V^i is of finite dimension and d is decomposable; that is, $d(V) \subset \Lambda^{\geq 2} V$. Moreover, $(\Lambda V, d)$ is equipped with a quasi-isomorphism $(\Lambda V, d) \xrightarrow{\sim} A_{\text{PL}}(X)$ to the commutative differential graded algebra $A_{\text{PL}}(X)$ of differential polynomial forms on X . Observe that, as algebras, $H^*(\Lambda V, d) \cong H^*(A_{\text{PL}}(X)) \cong H^*(X; \mathbb{Q})$.

Let M be a simply-connected \mathbb{Q} -Gorenstein space of dimension d and denote by $\rho : A = \Lambda V \rightarrow A_{\text{PL}}(M)$ a minimal Sullivan model for M . Then, by Theorem 2.2,

$$H^d(\text{Hom}_{A^{\otimes 2}}(\mathbb{B}, A^{\otimes 2})) \cong \text{Ext}_{A^{\otimes 2}}^d(A, A^{\otimes 2}) \cong H^0(M) = \mathbb{K},$$

where $\varepsilon : \mathbb{B} = B(A, A, A) \rightarrow A$ is the two-sided bar resolution of A ([8, Definition 5.51]), and denote by

$$\Delta^! : \mathbb{B} \longrightarrow A^{\otimes 2}$$

the map of right $A^{\otimes 2}$ -modules which corresponds to a generator of $H^0(M) = \mathbb{K}$. We will define the dual loop product and coproduct in §2 and §3 by using $\Delta^!$.

We conclude this section by recalling the map which is called the Eilenberg-Moore map in rational coefficient case. Consider a pull-back diagram

$$\begin{array}{ccc} E_f & \xrightarrow{\tilde{f}} & E \\ q \downarrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

in which $p : E \rightarrow B$ is a fibration and B is a simply-connected space. The induced map $f^* : A_{\text{PL}}(B) \rightarrow A_{\text{PL}}(X)$ gives $A_{\text{PL}}(X)$ a right $A_{\text{PL}}(B)$ -module structure and let $\tilde{\varepsilon} : P \rightarrow A_{\text{PL}}(X)$ be a semifree resolution as left $A_{\text{PL}}(B)$ -modules. Then, we have the quasi-isomorphism called the Eilenberg-Moore map

$$\text{EM} : P \otimes_{A_{\text{PL}}(B)} A_{\text{PL}}(E) \longrightarrow A_{\text{PL}}(E_f)$$

defined by $\text{EM}(u \otimes x) = q^* \tilde{\varepsilon}(u) \cdot \tilde{f}^*(x)$ for $u \otimes x \in P \otimes_{A_{\text{PL}}(B)} A_{\text{PL}}(E)$. For details, see [13], [15] for example.

3. DUAL LOOP PRODUCT AND ASSOCIATIVITY

In this section, we first recall the definition of the dual loop product on a simply-connected Gorenstein space M . Denote by $\rho : A \rightarrow A_{\text{PL}}(M)$ a minimal Sullivan model for M . The pull-back diagram

$$\begin{array}{ccc} LM \times_M LM & \xrightarrow{\text{inc}} & LM \times LM \\ \text{ev}_0 \downarrow & & \downarrow \text{ev}_0 \times \text{ev}_0 \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

gives rise to the Eilenberg-Moore map $\text{EM} : \mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}(LM \times LM) \rightarrow A_{\text{PL}}(LM \times_M LM)$. The *dual loop product* Dlp is the composite map

$$\begin{aligned} Dlp : H^*(LM) &\xrightarrow{H(\text{Comp})} H^*(LM \times_M LM) \\ &\cong \downarrow \text{EM}^{-1} \\ &H^*(\mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}(LM \times LM)) \xrightarrow{H(\Delta^! \otimes 1)} H^*(LM \times LM) \end{aligned}$$

Here $\Delta^!$ is the map stated in §2 and the map $\text{Comp} : LM \times_M LM \rightarrow LM$ is the concatenation of loops, namely,

$$\text{Comp}(\gamma_1, \gamma_2)(t) = \begin{cases} \gamma_1(2t) & (0 \leq t \leq \frac{1}{2}) \\ \gamma_2(2t-1) & (\frac{1}{2} \leq t \leq 1) \end{cases}$$

for $(\gamma_1, \gamma_2) \in LM \times_M LM$ and $t \in [0, 1]$.

Proposition 3.1. *The dual loop product satisfy the identity*

$$(Dlp \otimes 1)Dlp = (-1)^d(1 \otimes Dlp)Dlp.$$

Proof. We first consider the commutative diagram:

$$\begin{array}{ccccc} LM \times_M LM \times_M LM & \xrightarrow{\text{inc}} & LM^{\times 3} \\ \downarrow \text{ev}_0 & & \downarrow \text{ev}_0^{\times 3} \\ M & \xrightarrow{(\Delta \times 1)\Delta} & M^{\times 3} \\ \downarrow = & & \downarrow = \\ LM \times_M LM \times_M LM & \xrightarrow{\text{inc}} & (LM \times_M LM) \times LM & \xrightarrow{\text{inc}} & LM^{\times 3} \\ \downarrow \text{ev}_0 & \searrow \text{ev}_0 \times \text{ev}_0 & \downarrow \text{ev}_0^{\times 3} & & \downarrow \text{ev}_0^{\times 3} \\ M & \xrightarrow{\Delta} & M^{\times 2} & \xrightarrow{\Delta \times 1} & M^{\times 3} \end{array}$$

It is readily seen that the two bottom squares and the top square are pull-back diagrams and we denote by

$$\begin{aligned} \text{EM}_1 &: (\mathbb{B} \otimes A) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3}) \longrightarrow A_{\text{PL}}((LM \times_M LM) \times LM), \\ \text{EM}_2 &: \mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}((LM \times_M LM) \times LM) \longrightarrow A_{\text{PL}}(LM \times_M LM \times_M LM), \\ \text{EM}_3 &: (\mathbb{B} \otimes_A \mathbb{B}) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3}) \longrightarrow A_{\text{PL}}(LM \times_M LM \times_M LM) \end{aligned}$$

the Eilenberg-Moore maps of the diagram, respectively. We here remark that $\varepsilon \otimes 1 : \mathbb{B} \otimes A \rightarrow A \otimes A$ is a semifree resolution of $A^{\otimes 2}$ as $A^{\otimes 3}$ -modules with the $A^{\otimes 3}$ -module structure of $A^{\otimes 2}$ given by (product) $\otimes 1 : A^{\otimes 3} \rightarrow A^{\otimes 2}$, and $\varepsilon \cdot \varepsilon : \mathbb{B} \otimes_A \mathbb{B} \rightarrow A$ is a semifree resolution of A as $A^{\otimes 3}$ -modules. Then we have the following commutative diagram:

$$(3.1) \quad \begin{array}{ccc} H^*(LM) & \xrightarrow{=} & H^*(LM) \\ \downarrow H(\text{Comp}) & \xrightarrow{(I) \quad H(\text{Comp} \times 1)} & \downarrow H(\text{Comp}') \\ H^*(LM \times_M LM) & \xrightarrow{H(\text{Comp} \times 1)} & H^*(LM \times_M LM \times_M LM) \\ \downarrow \text{EM}_1^{-1} \cong & \xrightarrow{(II) \quad H(1 \otimes (\text{Comp} \times 1)^*)} & \downarrow \cong \text{EM}_2^{-1} \\ H^*(\mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}(LM^{\times 2})) & \xrightarrow{H(1 \otimes (\text{Comp} \times 1)^*)} & H^*(\mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}((LM \times_M LM) \times LM)) \\ \downarrow H(\Delta^! \otimes 1) & \searrow \text{(III) } H(\Delta^! \otimes 1) & \downarrow \cong (1 \otimes \text{EM}_1)^{-1} \\ H^*(LM^{\times 2}) & & H^*(\mathbb{B} \otimes_{A^{\otimes 2}} ((\mathbb{B} \otimes A) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3}))) \\ \downarrow H(\text{Comp} \times 1) & \swarrow \text{(IV) } H(\Delta^! \otimes 1) & \downarrow \cong \Gamma \\ H^*((LM \times_M LM) \times LM) & & H^*(\mathbb{B} \otimes_{A^{\otimes 2}} ((\mathbb{B} \otimes A) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3}))) \\ \downarrow \text{EM}_1^{-1} \cong & \swarrow \text{(V) } H(\Delta^! \otimes (1 \otimes 1) \otimes 1) & \downarrow \cong \Gamma \\ H^*(\mathbb{B} \otimes_A A) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3}) & & H^*(\mathbb{B} \otimes_A \mathbb{B}) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3}) \\ \downarrow H((\Delta^! \otimes 1) \otimes 1) & \xleftarrow{H((\Delta^! \otimes \Delta^!) \otimes 1)} & \downarrow \\ H^*(LM^{\times 3}) & & H^*(LM^{\times 3}) \end{array}$$

Here, the map $\text{Comp}' : LM \times_M LM \times_M LM \rightarrow LM$ is the composition map given by

$$\text{Comp}'(\gamma_1, \gamma_2, \gamma_3)(t) = \begin{cases} \gamma_1(3t) & (0 \leq t \leq \frac{1}{3}) \\ \gamma_2(3t-1) & (\frac{1}{3} \leq t \leq \frac{2}{3}) \\ \gamma_3(3t-2) & (\frac{2}{3} \leq t \leq 1). \end{cases}$$

The left $A^{\otimes 2}$ -module structure of $(\mathbb{B} \otimes A) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3})$ given by the natural right $A^{\otimes 2}$ -module structure of $\mathbb{B} \otimes A$, that is,

$$(b_1 \otimes b_2) \cdot ((u \otimes a) \otimes x) = (-1)^{|b_1||u|+|b_2|(|u|+|a|)}(ub_1 \otimes ab_2) \otimes x$$

for $(u \otimes a) \otimes x \in (\mathbb{B} \otimes A) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3})$ and $b_1 \otimes b_2 \in A^{\otimes 2}$, makes EM_1 a left $A^{\otimes 2}$ -module map. The chain map Γ is defined by

$$\Gamma(v \otimes (u \otimes a) \otimes x) = (-1)^{|u||v|}(u \otimes va) \otimes x$$

for $v \otimes (u \otimes a) \otimes x$ in $\mathbb{B} \otimes_{A^{\otimes 2}} ((\mathbb{B} \otimes A) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3}))$, and we also see that Γ is an isomorphism with the inverse map $\Gamma^{-1}((u \otimes v) \otimes x) = (-1)^{|u||v|}v \otimes (u \otimes 1) \otimes x$.

We now check the commutativity of the diagram (3.1). Since the composite $\text{Comp}(\text{Comp} \times 1)$ is homotopic to Comp' , the square (I) commutes. The equation (II) : $H(\text{Comp} \times 1)\text{EM} = \text{EM}_2 H(1 \otimes (\text{Comp} \times 1)^*)$ is shown by the naturality of the Eilenberg-Moore map and the commutative diagram:

$$\begin{array}{ccccc} LM \times_M LM \times_M LM & \xrightarrow{\text{inc}} & (LM \times_M LM) \times LM \\ \downarrow \text{ev}_0 & \downarrow \text{Comp} \times 1 & \downarrow \text{ev}_0 \times \text{ev}_0 & \downarrow \text{Comp} \times 1 & \downarrow \\ M & \xrightarrow{\Delta} & M^{\times 2} & \xrightarrow{\text{Comp} \times 1} & LM \times LM \\ \downarrow = & \downarrow \text{ev}_0 & \downarrow = & \downarrow \text{ev}_0^{\times 2} & \downarrow \\ M & \xrightarrow{\Delta} & M^{\times 2} & \xrightarrow{\text{Comp} \times 1} & LM \times LM \end{array}$$

A straightforward calculation shows a commutativity of the other squares (III), (IV) and (V), and also shows that the equality $\text{EM}_3 = \text{EM}_2(\text{EM}_1 \otimes 1)\Gamma^{-1}$. By the definition of dual loop products,

$$(Dlp \otimes 1)Dlp = H((\Delta^! \otimes 1) \otimes 1)\text{EM}_1^{-1}H(\text{Comp} \times 1)H(\Delta^! \otimes 1)\text{EM}_1^{-1}H(\text{Comp})$$

holds. Therefore, the commutativity of the diagram (3.1) implies that

$$(Dlp \otimes 1)Dlp = H((\Delta^! \otimes \Delta^!) \otimes 1)\text{EM}_3^{-1}H(\text{Comp}').$$

We next observe the composite $(1 \otimes Dlp)Dlp$. Consider the commutative diagram

$$\begin{array}{ccccc} LM \times_M LM \times_M LM & \xrightarrow{\text{inc}} & LM^{\times 3} \\ \downarrow \text{ev}_0 & \downarrow = & \downarrow \text{ev}_0^{\times 3} & \downarrow = & \downarrow \\ M & \xrightarrow{(1 \times \Delta)\Delta} & M^{\times 3} & \xrightarrow{\text{Comp} \times 1} & LM^{\times 3} \\ \downarrow = & \downarrow \text{ev}_0 & \downarrow \text{ev}_0 \times \text{ev}_0 & \downarrow \text{ev}_0^{\times 3} & \downarrow \\ M & \xrightarrow{\Delta} & M^{\times 2} & \xrightarrow{1 \times \Delta} & M^{\times 3} \end{array}$$

where the two bottom squares and the top square are pull-back diagrams, and denote by

$$\text{EM}_4 : (A \otimes \mathbb{B}) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3}) \longrightarrow A_{\text{PL}}(LM \times (LM \times_M LM)),$$

$$\text{EM}_5 : \mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}(LM \times (LM \times_M LM)) \longrightarrow A_{\text{PL}}(LM \times_M LM \times_M LM)$$

the Eilenberg-Moore map of the two bottom squares. Then, we have the commutative diagram:

$$\begin{array}{ccc}
 H^*(LM) & \xrightarrow{=} & H^*(LM) \\
 H(\text{Comp}) \downarrow & \xrightarrow{(I')} & \downarrow H(\text{Comp}') \\
 H^*(LM \times_M LM) & \xrightarrow{H(1 \times \text{Comp})} & H^*(LM \times_M LM \times_M LM) \\
 EM^{-1} \cong \downarrow & \xrightarrow{(II')} & \downarrow \cong EM_5^{-1} \\
 H^*(\mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}(LM^{\times 2})) & \xrightarrow{H(1 \otimes (1 \times \text{Comp})^*)} & H^*(\mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}(LM \times (LM \times_M LM))) \\
 H(\Delta^! \otimes 1) \downarrow & \xrightarrow{(III')} & \downarrow \cong (1 \otimes EM_4)^{-1} \\
 H^*(LM^{\times 2}) & \xrightarrow{H(\Delta^! \otimes 1)} & H^*(\mathbb{B} \otimes_{A^{\otimes 2}} ((A \otimes \mathbb{B}) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3}))) \\
 H(1 \times \text{Comp}) \downarrow & \xrightarrow{(IV')} & \downarrow \Gamma' \\
 H^*(LM \times (LM \times_M LM)) & \xrightarrow{H(\Delta^! \otimes (1 \otimes 1) \otimes 1)} & H^*(\mathbb{B} \otimes_{A^{\otimes 2}} ((A \otimes \mathbb{B}) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3}))) \\
 EM_4^{-1} \cong \downarrow & \xrightarrow{(V')} & \downarrow \\
 H^*((A \otimes \mathbb{B}) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3})) & \xrightarrow{(-1)^d H((\Delta^! \otimes \Delta^!) \otimes 1)} & H^*((\mathbb{B} \otimes_A \mathbb{B}) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3})). \\
 H((1 \otimes \Delta^!) \otimes 1) \downarrow & &
 \end{array}$$

Remark that the left $A^{\otimes 2}$ -module structure of $(A \otimes \mathbb{B}) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3})$ given by the natural left $A^{\otimes 2}$ -module structure of $A \otimes \mathbb{B}$, that is,

$$(b_1 \otimes b_2) \cdot ((a \otimes u) \otimes x) = (-1)^{|a||b_2|} (b_1 a \otimes b_2 u) \otimes x$$

for $(a \otimes u) \otimes x \in (A \otimes \mathbb{B}) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3})$ and $b_1 \otimes b_2 \in A^{\otimes 2}$. It is readily seen that the map EM'_1 preserves the left $A^{\otimes 2}$ -module structure. The chain map Γ' is defined by $\Gamma'(v \otimes (a \otimes u) \otimes x) = (-1)^{|a||v|} (av \otimes u) \otimes x$ for any $v \otimes (a \otimes u) \otimes x$ in $\mathbb{B} \otimes_{A^{\otimes 2}} ((A \otimes \mathbb{B}) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 3}))$.

We can check the commutativity of the diagram (3.2) as with the proof of the commutativity of the diagram (3.1) described above. Indeed, we see that

$$(1 \otimes Dlp)Dlp = H((1 \otimes \Delta^!) \otimes 1)EM_4^{-1}H(1 \times \text{Comp})H(\Delta^! \otimes 1)EM^{-1}H(\text{Comp}),$$

and the square (I') commutes since $(1 \times \text{Comp})\text{Comp}$ is homotopic to Comp' . The equality (II') : $H(1 \times \text{Comp})EM = EM_5 H(1 \otimes (1 \times \text{Comp})^*)$ is shown by the naturality of the Eilenberg-Moore map and the following commutative diagram:

$$\begin{array}{ccccc}
 LM \times_M LM \times_M LM & \xrightarrow{\text{inc}} & LM \times (LM \times_M LM) \\
 \downarrow \text{ev}_0 & \searrow \Delta & \downarrow 1 \times \text{Comp} & \swarrow \text{ev}_0 \times \text{ev}_0 & \downarrow 1 \times \text{Comp} \\
 M & \xrightarrow{\Delta} & M^{\times 2} & \xrightarrow{=} & LM \times LM \\
 \downarrow = & \swarrow \text{ev}_0 & \downarrow \text{inc} & \swarrow \text{ev}_0^{\times 2} & \\
 M & \xrightarrow{\Delta} & M^{\times 2} & \xrightarrow{=} &
 \end{array}$$

We also see that the commutativity of the diagrams (III'), (IV'), (V) and the equality $EM_3 = EM_5(EM_4 \otimes 1)(\Gamma')^{-1}$ by a straightforward calculation. Therefore, conclude that

$$(-1)^d(1 \otimes Dlp)Dlp = H((\Delta^! \otimes \Delta^!) \otimes 1)EM_3^{-1}H(\text{Comp}') = (Dlp \otimes 1)Dlp.$$

□

4. DUAL LOOP COPRODUCT AND ASSOCIATIVITY

We begin recalling the definition of the dual loop coproduct on Gorenstein spaces. Consider the pull-back diagram

$$\begin{array}{ccc} LM \times_M LM & \xrightarrow{\text{Comp}} & LM \\ \text{ev}_0 \downarrow & & \downarrow j \\ M & \xrightarrow{\Delta} & M \times M, \end{array}$$

where the map j is defined by $j(\gamma) = (\gamma(0), \gamma(\frac{1}{2}))$ for any loop γ on M . It gives rise to the Eilenberg-Moore map

$$\text{EM}' : \mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}(LM) \longrightarrow A_{\text{PL}}(LM \times_M LM)$$

and we have the composition map called the *dual loop coproduct*

$$\begin{aligned} \text{Dlco} : H^*(LM \times LM) &\xrightarrow{H(\text{inc})} H^*(LM \times_M LM) \\ &\cong \downarrow (\text{EM}')^{-1} \\ &H^*(\mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}(LM)) \xrightarrow{H(\Delta^! \otimes 1)} H^*(LM). \end{aligned}$$

Proposition 4.1. *The dual loop coproduct satisfy the identity*

$$\text{Dlco}(\text{Dlco} \otimes 1) = (-1)^d \text{Dlco}(1 \otimes \text{Dlco}).$$

Proof. We first consider the following pull-back diagram:

$$(4.1) \quad \begin{array}{ccc} LM \times_M LM & \xrightarrow{\text{Comp}} & LM \\ \text{ev}_0 \downarrow & & \downarrow T_M \circ j \\ M & \xrightarrow{\Delta} & M \times M, \end{array}$$

where $T_M : M^{\times 2} \rightarrow M^{\times 2}$ is the switching map. Let $\tilde{\text{EM}}'$ be the Eilenberg-Moore map of the diagram (4.1), $\tau : A^{\otimes 2} \rightarrow A^{\otimes 2}$ the switching map and $\tilde{\tau} : \mathbb{B} \rightarrow \mathbb{B}$ a chain map satisfy $\tilde{\tau}^2 = 1$ and makes the diagram commutes:

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\tilde{\tau}} & \mathbb{B} \\ \iota \uparrow & & \uparrow \iota \\ A^{\otimes 2} & \xrightarrow{\tau} & A^{\otimes 2}, \end{array}$$

where ι is the inclusion. Both the map $\Delta^!$ and $\tau \circ \Delta^! \circ \tilde{\tau}$ are in $\text{Ext}_{A^{\otimes 2}}^d(A, A^{\otimes 2}) \cong \mathbb{Q}$, there is a scalar $\lambda \in \mathbb{Q}$ such that $\Delta^! = \lambda \tau \circ \Delta^! \circ \tilde{\tau}$ in $\text{Ext}_{A^{\otimes 2}}^d(A, A^{\otimes 2})$. Hence, we obtain the following commutative diagram:

$$\begin{array}{ccc} & H^*(LM \times LM) & \\ & \downarrow H(\text{inc}) & \\ & H^*(LM \times_M LM) & \\ & \downarrow (\text{EM}')^{-1} \cong & \\ & H^*(\mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}(LM)) & \xrightarrow{\tilde{\tau} \otimes 1} H^*(\mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}(LM)) \\ & \downarrow H(\Delta^! \otimes 1) & \nwarrow \lambda H(\Delta^! \otimes 1) \\ & H^*(LM \times LM). & \end{array}$$

Dlco (curved arrow from $H^*(LM \times LM)$ to $H^*(LM \times LM)$)

Here, the commutativity of the upper right triangle is shown by the commutative diagram:

$$\begin{array}{ccccc}
 & LM \times_M LM & \xrightarrow{\text{Comp}} & LM & \\
 \swarrow ev_0 & \downarrow = & & \swarrow j & \downarrow = \\
 M & \xrightarrow{\Delta} & M^{\times 2} & & \\
 \downarrow = & \swarrow ev_0 & \downarrow = & \downarrow T_M & \\
 & LM \times_M LM & \xrightarrow{\text{Comp}} & LM & \\
 & \downarrow = & & \swarrow T_M \circ j & \\
 M & \xrightarrow{\Delta} & M^{\times 2} & &
 \end{array}$$

Since $\Delta^! = \tau\tau\Delta^!\tilde{\tau}\tilde{\tau} = \lambda\tau\Delta^!\tilde{\tau} = \lambda^2\Delta^!$, we see that $\lambda^2 = 1$. Let EM'_1 and $\tilde{\text{EM}}'_1$ be the Eilenberg-Moore maps of the top and bottom square of the following diagram, respectively:

$$(4.2) \quad
 \begin{array}{ccccc}
 (LM \times_M LM) \times LM & \xrightarrow{\text{Comp} \times 1} & LM \times LM & & \\
 \swarrow ev_0 \times ev_0 & \downarrow = & \swarrow j \times ev_0 & \downarrow = & \\
 M^{\times 2} & \xrightarrow{\Delta \times 1} & M^{\times 3} & & \\
 \downarrow = & \swarrow ev_0 \times ev_0 & \downarrow = & \downarrow T_M \times 1 & \\
 & (LM \times_M LM) \times LM & \xrightarrow{\text{Comp} \times 1} & LM \times LM & \\
 & \downarrow = & & \swarrow T_M \circ j \times ev_0 & \\
 M^{\times 2} & \xrightarrow{\Delta \times 1} & M^{\times 3} & &
 \end{array}$$

Then, the definition of the dual loop coproduct and the equation $\lambda^2 = 1$ show that

$$\begin{aligned}
 \text{Dl} \text{cop}(\text{Dl} \text{cop} \otimes 1) &= H(\Delta^! \otimes 1)(\text{EM}')^{-1}H(\text{inc})H((\Delta^! \otimes 1) \otimes 1)(\text{EM}'_1)^{-1}H(\text{inc} \times 1) \\
 &= \lambda^2 H(\Delta^! \otimes 1)(\tilde{\text{EM}}')^{-1}H(\text{inc})H((\Delta^! \otimes 1) \otimes 1)(\tilde{\text{EM}}'_1)^{-1}H(\text{inc} \times 1) \\
 &= H(\Delta^! \otimes 1)(\tilde{\text{EM}}')^{-1}H(\text{inc})H((\Delta^! \otimes 1) \otimes 1)(\tilde{\text{EM}}'_1)^{-1}H(\text{inc} \times 1).
 \end{aligned}$$

Consider the commutative diagram

$$(4.3) \quad
 \begin{array}{ccccccc}
 & LM \times_M LM \times_M LM & \xrightarrow{\text{Comp}'} & LM & & & \\
 \swarrow ev_0 & \downarrow = & & \swarrow j' & \downarrow \xi_1 & & \\
 M & \xrightarrow{\Delta} & M^{\times 3} & & & & \\
 \downarrow = & \swarrow ev_0 & \downarrow = & \downarrow j_2 & \downarrow j_1 & & \\
 & LM \times_M LM \times_M LM & \xrightarrow{\text{Comp} \times 1} & LM \times_M LM & \xrightarrow{\text{Comp}} & LM & \\
 & \downarrow = & & \downarrow = & & & \\
 M & \xrightarrow{\Delta} & M \times M & \xrightarrow{1 \times \Delta} & M^{\times 3} & &
 \end{array}$$

where the maps j' , j_1 and j_2 are defined by

$$j'(\gamma) = (\gamma\left(\frac{1}{3}\right), \gamma(0), \gamma\left(\frac{2}{3}\right)), \quad j_1 = (\gamma\left(\frac{1}{4}\right), \gamma(0), \gamma\left(\frac{1}{2}\right)), \quad j_2(\gamma_1, \gamma_2) = (\gamma_1\left(\frac{1}{2}\right), \gamma_1(0))$$

for any $\gamma \in LM$ and $(\gamma_1, \gamma_2) \in LM \times_M LM$, respectively. The map ξ_1 is given by

$$\xi_1(\gamma)(t) = \begin{cases} \gamma\left(\frac{3}{4}t\right) & (0 \leq t \leq \frac{2}{3}) \\ \gamma\left(\frac{3}{2}t - \frac{1}{2}\right) & (\frac{2}{3} \leq t \leq 1) \end{cases}$$

for $\gamma \in LM$ and $t \in [0, 1]$. Since the bottom two squares and the top square of the diagram (4.3) are pull-back diagrams, it give rise to the Eilenberg-Moore maps

$$EM'_2 : \mathbb{B} \otimes_{A \otimes 2} A_{PL}(LM \times_M LM) \longrightarrow A_{PL}(LM \times_M LM \times_M LM),$$

$$EM'_3 : (A \otimes \mathbb{B}) \otimes_{A \otimes 3} A_{PL}(LM) \longrightarrow A_{PL}(LM \times_M LM),$$

$$EM'_4 : (\mathbb{B} \otimes_A \mathbb{B}) \otimes_{A \otimes 3} A_{PL}(LM) \longrightarrow A_{PL}(LM \times_M LM \times_M LM),$$

respectively. We here note that there is a map between the bottom and left-hand side square of the diagram (4.3) and the diagram (4.2) such as the following:

$$(4.4) \quad \begin{array}{ccccc} & LM \times_M LM \times_M LM & \xrightarrow{\text{Comp} \times 1} & LM \times_M LM & \\ \swarrow \text{ev}_0 & \downarrow \text{inc} & & \swarrow j_2 & \downarrow \text{inc} \\ M & \xrightarrow{\Delta} & M \times M & & \\ \downarrow \Delta & \swarrow \text{ev}_0 \times \text{ev}_0 & \downarrow 1 \times \Delta & \swarrow T_M \circ j \times \text{ev}_0 & \\ M \times M & \xrightarrow{\Delta \times 1} & M^{\times 3} & & \end{array}$$

We hence get the following commutative diagram:

$$(4.5) \quad \begin{array}{ccc} H^*(LM \times LM \times LM) & \xrightarrow{H(\text{inc})} & H^*(LM \times_M LM \times_M LM) \\ \downarrow H(\text{inc} \times 1) & \text{(I)} & \downarrow H(\text{inc}) \\ H^*((LM \times_M LM) \times LM) & \xrightarrow{H(\text{inc})} & H^*(LM \times_M LM \times_M LM) \\ \downarrow (\tilde{EM}'_1)^{-1} \cong & \text{(II)} & \downarrow (\tilde{EM}'_2)^{-1} \cong \\ H^*((\mathbb{B} \otimes A) \otimes_{A \otimes 3} A_{PL}(LM \times LM)) & \xrightarrow{\Delta^* \otimes (1 \times \Delta)^* \text{inc}^*} & H^*(\mathbb{B} \otimes_{A \otimes 2} A_{PL}(LM \times_M LM)) \\ \downarrow H((\Delta^! \otimes 1) \otimes 1) & \text{(III)} & \downarrow 1 \otimes (\tilde{EM}'_3)^{-1} \cong \\ H^*(LM \times LM) & \xrightarrow{H(\Delta^! \otimes 1)} & H^*(LM \times_M LM) \\ \downarrow H(\text{inc}) & \text{(IV)} & \downarrow (\tilde{EM}')^{-1} \cong \\ H^*(LM \times_M LM) & \xleftarrow{\Theta'_1} & H^*(\mathbb{B} \otimes_{A \otimes 2} ((A \otimes \mathbb{B}) \otimes_{A \otimes 3} A_{PL}(LM))) \\ \downarrow H(\Delta^! \otimes 1) & \text{(V)} & \downarrow \Theta_1 \\ H^*(LM) & \xleftarrow{(-1)^d H(\xi_1) H((\Delta^! \otimes \Delta^!) \otimes 1)} & H^*(\mathbb{B} \otimes_A \mathbb{B}) \otimes_{A \otimes 3} A_{PL}(LM) \end{array}$$

Here, $\Delta^* \otimes_{(1 \times \Delta)^*} \text{inc}^*$ is given by

$$(\Delta^* \otimes_{(1 \times \Delta)^*} \text{inc}^*)((u \otimes a) \otimes x) = ua \otimes \text{inc}^*(x)$$

for $(u \otimes a) \otimes x \in (\mathbb{B} \otimes A) \otimes_{A \otimes 3} A_{PL}(LM \times LM)$. The chain maps Θ_1 and Θ'_1 are defined as follows:

$$\Theta_1((u \otimes v) \otimes x) = u \otimes (1 \otimes v) \otimes \xi_1^*(x),$$

$$\Theta'_1(v \otimes (a \otimes u) \otimes x) = \sum (-1)^{|v_2||a| + (|v_1| + |a|)(|v_2| + |u|)} v_2 u \otimes \left(\text{ev}_{\frac{1}{4}}^*(v_1 a) \cdot x \right),$$

where $\Delta^!(v) = \sum v_1 \otimes v_2$. We also note that the left $A^{\otimes 2}$ -module structure of $(A \otimes \mathbb{B}) \otimes_{A \otimes 3} A_{PL}(LM)$ given by the natural left $A^{\otimes 2}$ -module structure of $A \otimes \mathbb{B}$ makes EM'_3 a left $A^{\otimes 2}$ -module map.

We now check the commutativity of the diagram (4.5). A commutativity of (I)

is trivial, and (II) is shown by the naturality of the Eilenberg-Moore map and the commutative diagram (4.4). It is readily seen that the diagram (III), (IV) and (V). By the diagram (4.3), we see that $EM'_4 = EM'_2(1 \otimes EM'_3)\Theta_1$. Therefore,

$$Dl\text{cop}(Dl\text{cop} \otimes 1) = (-1)^d H(\xi_1) H((\Delta^! \otimes \Delta^!) \otimes 1) (EM'_4)^{-1} H(\text{inc}).$$

On the other hand, consider the two commutative diagram

$$(4.6) \quad \begin{array}{ccccc} & LM \times_M LM \times_M LM & \xrightarrow{1 \times \text{Comp}} & LM \times_M LM & \\ \swarrow ev_0 & \downarrow \text{inc} & & \swarrow j'_2 & \downarrow \text{inc} \\ M & \xrightarrow{\Delta} & M \times M & & \\ \downarrow \Delta & \swarrow ev_0 \times ev_0 & \downarrow \Delta \times 1 & \swarrow ev_0 \times j & \\ M \times M & \xrightarrow{1 \times \Delta} & M \times M & \xrightarrow{1 \times \text{Comp}} & LM \times LM \end{array}$$

and

$$(4.7) \quad \begin{array}{ccccc} & LM \times_M LM \times_M LM & \xrightarrow{\text{Comp}'} & LM & \\ \swarrow ev_0 & \uparrow & & \swarrow j' & \uparrow \xi_2 \\ M & \xrightarrow{\Delta} & M \times M & \xrightarrow{\Delta \times 1} & M \times M \\ \uparrow = & \downarrow = & \downarrow j'_2 & \downarrow j'_1 & \\ M & \xrightarrow{\Delta} & M \times M & \xrightarrow{\Delta \times 1} & M \times M \end{array}$$

where j'_1 and j'_2 are defined by $j'_1(\gamma) = (\gamma(\frac{1}{2}), \gamma(0), \gamma(\frac{3}{4}))$, $j'_2(\gamma_1, \gamma_2) = (\gamma_2(0), \gamma_2(\frac{1}{2}))$ for $\gamma \in LM$ and $(\gamma_1, \gamma_2) \in LM \times_M LM$, respectively. The map ξ_2 is given by

$$\xi_2(\gamma)(t) = \begin{cases} \gamma(\frac{3}{2}t) & (0 \leq t \leq \frac{1}{3}) \\ \gamma(\frac{3}{4}t + \frac{1}{4}) & (\frac{1}{3} \leq t \leq 1). \end{cases}$$

We denote by

$$\begin{aligned} EM'_5 &: (A \otimes \mathbb{B}) \otimes_{A \otimes \mathbb{B}} A_{\text{PL}}(LM \times LM) \longrightarrow A_{\text{PL}}(LM \times (LM \times_M LM)), \\ EM'_6 &: \mathbb{B} \otimes_{A \otimes \mathbb{B}} A_{\text{PL}}(LM \times_M LM) \longrightarrow A_{\text{PL}}(LM \times_M LM \times_M LM) \\ EM'_7 &: (\mathbb{B} \otimes A) \otimes_{A \otimes \mathbb{B}} A_{\text{PL}}(LM) \longrightarrow A_{\text{PL}}(LM \times_M LM) \end{aligned}$$

the Eilenberg-Moore maps of the bottom square of the diagram (4.6), the top square of (4.6) and the bottom and right-hand side square of (4.7), respectively. Then, we

have the commutative diagram:

$$\begin{array}{ccc}
 H^*(LM \times LM \times LM) & & \\
 \downarrow H(1 \times \text{inc}) & \searrow H(\text{inc}) & \\
 H^*(LM \times (LM \times_M LM)) & \xrightarrow{H(\text{inc})} & H^*(LM \times_M LM \times_M LM) \\
 \downarrow (EM'_5)^{-1} \cong & & \downarrow (EM'_6)^{-1} \cong \\
 H^*((A \otimes \mathbb{B}) \otimes_{A \otimes 3} A_{PL}(LM \times LM)) & \xrightarrow{\Delta^* \otimes (\Delta \times 1)^* \text{inc}^*} & H^*(\mathbb{B} \otimes_{A \otimes 2} A_{PL}(LM \times_M LM)) \\
 \downarrow H((1 \otimes \Delta^!) \otimes 1) & \searrow H(\Delta^! \otimes 1) & \downarrow \cong 1 \otimes (EM'_7)^{-1} \\
 H^*(LM \times LM) & & \\
 \downarrow H(\text{inc}) & & \\
 H^*(LM \times_M LM) & & \\
 \downarrow (EM')^{-1} \cong & \xleftarrow{\Theta'_2} & H^*(\mathbb{B} \otimes_{A \otimes 2} ((\mathbb{B} \otimes A) \otimes_{A \otimes 3} A_{PL}(LM))) \\
 H^*(\mathbb{B} \otimes_{A \otimes 2} A_{PL}(LM)) & \xleftarrow{\Theta'_2} & H^*(\mathbb{B} \otimes_{A \otimes 2} ((\mathbb{B} \otimes A) \otimes_{A \otimes 3} A_{PL}(LM))) \\
 \downarrow H(\Delta^! \otimes 1) & \xleftarrow{H(\xi_2)H((\Delta^! \otimes \Delta^!) \otimes 1)} & \uparrow \Theta_2 \\
 H^*(LM) & \xleftarrow{H(\xi_2)H((\Delta^! \otimes \Delta^!) \otimes 1)} & H^*((\mathbb{B} \otimes_A \mathbb{B}) \otimes_{A \otimes 3} A_{PL}(LM)).
 \end{array}$$

Here, $\Delta^* \otimes (\Delta \times 1)^* \text{inc}^*$ is the map induced by the map between the pull-back diagrams described in (4.7), that is,

$$(\Delta^* \otimes (\Delta \times 1)^* \text{inc}^*)((a \otimes u) \otimes x) = au \otimes \text{inc}^*(x)$$

for $(a \otimes u) \otimes x \in (A \otimes \mathbb{B}) \otimes_{A \otimes 3} A_{PL}(LM \times LM)$. The chain maps Θ_2 and Θ'_2 are given by

$$\begin{aligned}
 \Theta_2((u \otimes v) \otimes x) &= (-1)^{|u||v|} v \otimes (u \otimes 1) \otimes \xi_2^*(x), \\
 \Theta'_2(v \otimes (u \otimes a) \otimes x) &= \sum (-1)^{|u||v_1| + (|u| + |a|)|v_2|} uv_1 \otimes \left(ev_{\frac{3}{4}}^* \rho(av_2) \cdot x \right),
 \end{aligned}$$

where $\Delta^!(v) = \sum v_1 \otimes v_2$. By the definition of the dual loop coproduct, we see that

$$Dlcop(1 \otimes Dlcop) = H(\Delta^! \otimes 1)(EM')^{-1} H(\text{inc}) H((1 \otimes \Delta^!) \otimes 1)(EM'_5)^{-1} H(1 \times \text{inc})$$

Similar argument of the proof of the commutativity of the diagram (4.5) described above shows the commutativity of the diagram (4.8). We also see that the equation $EM'_4 = \Theta_2(1 \otimes EM'_7)EM'_6$ holds, therefore the commutativity of the diagram (4.8) shows

$$Dlcop(1 \otimes Dlcop) = H(\xi_2)H((\Delta^! \otimes \Delta^!) \otimes 1)EM'_4{}^{-1}H(\text{inc}).$$

Since the map $H : LM \times [0, 1] \rightarrow LM$ defined by

$$H(\gamma, s)(t) = \begin{cases} \gamma(\frac{3}{4}(1+s)t) & (0 \leq t \leq \frac{1}{3}) \\ \gamma(\frac{3}{4}t + \frac{1}{4}s) & (\frac{1}{3} \leq t \leq \frac{2}{3}) \\ \gamma((\frac{3}{2} - \frac{3}{4}s)t - \frac{1}{2} + \frac{3}{4}s) & (\frac{2}{3} \leq t \leq 1), \end{cases}$$

for $\gamma \in LM$ and $s, t \in [0, 1]$ is a homotopy from ξ_1 to ξ_2 , we have

$$Dlcop(Dlcop \otimes 1) = (-1)^d Dlcop(1 \otimes Dlcop).$$

This completes the proof. \square

5. FROBENIUS COMPATIBILITY

In this section, we prove that the dual loop product and the dual loop coproduct satisfy the Frobenius compatibility.

Proposition 5.1. *One has*

$$(-1)^d(1 \otimes Dlcop)(Dlp \otimes 1) = Dlp \circ Dlcop = (-1)^d(Dlcop \otimes 1)(1 \otimes Dlp).$$

Proof. We note that the notations EM_i and EM'_i are described in §2 and §3, respectively. Consider the following three commutative diagrams:

$$(5.1) \quad \begin{array}{ccccc} LM \times_M LM \times_M LM & \xrightarrow{\text{Comp} \times 1} & LM \times_M LM & & \\ \text{\scriptsize ev_0} \swarrow & \downarrow (\text{Comp} \times 1)T_{(123)} & \swarrow \text{\scriptsize $(ev_{1,0}, ev_{1,\frac{1}{2}})$} & & \\ M & \xrightarrow{\Delta} & M^{\times 2} & \xrightarrow{\xi \text{Comp}} & LM \\ \downarrow = & \downarrow \text{\scriptsize ev_0} & \downarrow \text{\scriptsize $Comp$} & \downarrow = & \downarrow \\ M & \xrightarrow{\Delta} & M^{\times 2} & \xrightarrow{j} & LM \end{array}$$

$$(5.2) \quad \begin{array}{ccccc} LM \times_M LM \times_M LM & \xrightarrow{\text{inc}} & LM \times (LM \times_M LM) & & \\ \text{\scriptsize ev_0} \swarrow & \downarrow \text{\scriptsize inc} & \swarrow \text{\scriptsize $ev_0 \times ev_0$} & & \\ M & \xrightarrow{\Delta} & M^{\times 2} & \xrightarrow{inc} & LM^{\times 3} \\ \downarrow \Delta & \downarrow \text{\scriptsize $(LM \times_M LM) \times LM$} & \downarrow \text{\scriptsize $1 \times \Delta$} & \downarrow \text{\scriptsize inc} & \downarrow \\ M \times M & \xrightarrow{\Delta \times 1} & M^{\times 3} & \xrightarrow{ev_0^{\times 3}} & LM^{\times 3} \end{array}$$

$$(5.3) \quad \begin{array}{ccccccc} LM \times_M LM \times_M LM & \xrightarrow{\text{inc}} & LM \times (LM \times_M LM) & \xrightarrow{1 \times \text{Comp}} & LM^{\times 2} & & \\ \text{\scriptsize ev_0} \swarrow & \downarrow T_{(321)} & \swarrow \text{\scriptsize $ev_0 \times ev_0$} & \downarrow \text{\scriptsize $1 \times \Delta$} & \downarrow \text{\scriptsize $ev_0 \times j$} & \downarrow = & \\ M & \xrightarrow{\Delta} & M^{\times 2} & \xrightarrow{1 \times \Delta} & M^{\times 3} & \xrightarrow{\quad} & LM^{\times 2} \\ \downarrow = & \downarrow \text{\scriptsize $T_{(321)}$} & \downarrow \text{\scriptsize $Comp \times 1$} & \downarrow \text{\scriptsize $(ev_{1,0}, ev_{1,\frac{1}{2}})$} & \downarrow \text{\scriptsize $(inc) \circ T$} & \downarrow = & \\ M & \xrightarrow{\Delta} & M^{\times 2} & \xrightarrow{\Delta \times 1} & M^{\times 3} & \xrightarrow{ev_0 \times j} & LM^{\times 2} \end{array}$$

Here $T : LM \times_M LM \rightarrow LM \times_M LM$ is the switching map and $\xi : LM \rightarrow LM$ is a map given by

$$\xi(\gamma)(t) = \begin{cases} \gamma(2t + \frac{1}{2}) & (0 \leq t \leq \frac{1}{4}) \\ \gamma(t - \frac{1}{4}) & (\frac{1}{4} \leq t \leq \frac{1}{2}) \\ \gamma(\frac{1}{2}t) & (\frac{1}{2} \leq t \leq 1) \end{cases}$$

for $\gamma \in LM$ and $t \in [0, 1]$, and the maps $T_{(123)}$ and $T_{(321)}$ are given by

$$T_{(123)}(\gamma_1, \gamma_2, \gamma_3) = (\gamma_3, \gamma_1, \gamma_2), \quad T_{(321)}(\gamma_1, \gamma_2, \gamma_3) = (\gamma_2, \gamma_3, \gamma_1)$$

for $(\gamma_1, \gamma_2, \gamma_3)$ in $LM \times_M LM \times_M LM$. Denote by

$$EM''_1 : \mathbb{B} \otimes_{A^{\otimes 2}} A_{PL}(LM \times_M LM) \rightarrow A_{PL}(LM \times_M LM \times_M LM)$$

$$EM''_2 : (\mathbb{B} \otimes A) \otimes_{A^{\otimes 3}} A_{PL}(LM^{\times 2}) \rightarrow A_{PL}(LM \times_M LM)$$

the Eilenberg-Moore maps of the top square of (5.1) and the bottom and right-hand side square of (5.3), respectively. In order to obtain the result, we consider the following diagrams:

(5.4)

$$\begin{array}{ccccccc}
& & & (Dlp \otimes 1) & & & \\
H^*(LM \times LM) & \xrightarrow{H(\text{Comp} \times 1)} & H^*((LM^{\times M^2}) \times LM) & \xrightarrow[\cong]{(EM_1)^{-1}} & H^*(\mathbb{B} \otimes A) \otimes_{A \otimes 3} A_{\text{PL}}(LM^{\times 3}) & \xrightarrow{H((\Delta^! \otimes 1) \otimes 1)} & H^*(LM^{\times 3}) \\
\downarrow H(\text{inc}) & \text{(I)} & \downarrow H(\text{inc}) & \text{(II)} & \downarrow \Delta^* \otimes (1 \times \Delta)^* \text{inc}^* & \text{(III)} & \downarrow H(1 \times \text{inc}) \\
H^*(LM \times_M LM) & \xrightarrow{H(\text{Comp} \times_M 1)} & H^*(LM^{\times M^3}) & \xrightarrow[\cong]{(EM_5)^{-1}} & H^*(\mathbb{B} \otimes_{A \otimes 2} A_{\text{PL}}(LM \times (LM^{\times M^2}))) & \xrightarrow{H(\Delta^! \otimes 1)} & H^*(LM \times (LM^{\times M^2})) \\
\downarrow \cong & & \uparrow \cong & & \downarrow & & \downarrow \cong \\
\cong & (EM')^{-1} & \text{(IV)} & H^*(LM^{\times M^3}) & \text{(V)} & H^*(\mathbb{B} \otimes_{A \otimes 2} ((A \otimes \mathbb{B}) \otimes_{A \otimes 3} A_{\text{PL}}(LM^{\times 2}))) & \xrightarrow{H(\Delta^! \otimes (1 \otimes 1) \otimes 1)} H^*((A \otimes \mathbb{B}) \otimes_{A \otimes 3} A_{\text{PL}}(LM^{\times 2})) \\
& & & \uparrow \cong & & \uparrow \cong & \uparrow \cong \\
& & & (EM'_1)^{-1} & & \Phi & \\
H^*(\mathbb{B} \otimes_{A \otimes 2} A_{\text{PL}}(LM)) & \xrightarrow{1 \otimes (\xi \text{Comp})^*} & H^*(\mathbb{B} \otimes_{A \otimes 2} A_{\text{PL}}(LM^{\times M^2})) & \xrightarrow[\cong]{(1 \otimes EM'_2)^{-1}} & H^*(\mathbb{B} \otimes_{A \otimes 2} ((\mathbb{B} \otimes A) \otimes_{A \otimes 3} A_{\text{PL}}(LM^{\times 2}))) & \xrightarrow{H((1 \otimes \Delta^!) \otimes 1)} & H^*(LM \times LM) \\
\downarrow H(\Delta^! \otimes 1) & \text{(VII)} & \downarrow H(T)H(\Delta^! \otimes 1) & \text{(VIII)} & \downarrow \Phi' & & \downarrow \\
H^*(LM) & \xrightarrow{H(\text{Comp})} & H^*(LM^{\times M^2}) & \xrightarrow[\cong]{(EM)^{-1}} & H^*(\mathbb{B} \otimes_{A \otimes 2} A_{\text{PL}}(LM^{\times 2})) & \xrightarrow{(-1)^d H(\Delta^! \otimes 1)} & H^*(LM \times LM)
\end{array}$$

\xrightarrow{Dlp}

Remark that the left $A^{\otimes 2}$ -module structure of $(A \otimes \mathbb{B}) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 2})$ given by the natural left $A^{\otimes 2}$ -module structure of $A \otimes \mathbb{B}$ makes EM_5 a left $A^{\otimes 2}$ -module map. Similarly, the left $A^{\otimes 2}$ -module structure of $(\mathbb{B} \otimes A) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 2})$ given by the natural right $A^{\otimes 2}$ -module structure of $\mathbb{B} \otimes A$ makes EM_2'' a left $A^{\otimes 2}$ -module map. The chain map Φ is defined by

$$\Phi(v \otimes (u \otimes a) \otimes x) = (-1)^{|u||v|} u \otimes (1 \otimes va) \otimes x$$

for $v \otimes (u \otimes a) \otimes x \in \mathbb{B} \otimes_{A^{\otimes 2}} ((\mathbb{B} \otimes A) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 2}))$. We also see that Φ is an isomorphism with the inverse map

$$\Phi^{-1}(v \otimes (a \otimes u) \otimes x) = (-1)^{|u|(|a|+|v|)+|a||v|} u \otimes (av \otimes 1) \otimes x$$

for $v \otimes (a \otimes u) \otimes x$ in $\mathbb{B} \otimes_{A^{\otimes 2}} ((A \otimes \mathbb{B}) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 2}))$. The chain map Φ' is also given by

$$\Phi'(v \otimes (u \otimes a) \otimes x) = \sum (-1)^{|u||v_1|+|u||v_2|+|a||v_2|} uv_1 \otimes \left(ev_{2,\frac{1}{2}}^* \rho(av_2) \cdot x \right)$$

for $v \otimes (u \otimes a) \otimes x \in \mathbb{B} \otimes_{A^{\otimes 2}} ((\mathbb{B} \otimes A) \otimes_{A^{\otimes 3}} A_{\text{PL}}(LM^{\times 2}))$, where $\Delta^!(v) = \sum v_1 \otimes v_2$. We here show the commutativity of the diagram (5.4). The commutativity of the square (I) is trivial. The naturality of the Eilenberg-Moore map and the commutative diagrams (5.1), (5.2) and (5.3) show that the diagrams (IV), (II) and (V) commute. The straightforward calculation show the commutativity of (III), (VI), (VIII) and (IX). Since ξ is homotopic to the identity map with the homotopy $H' : LM \times I \rightarrow LM$ given by

$$H'(\gamma, s)(t) = \begin{cases} \gamma(\frac{2}{2-s}t + \frac{4-3s}{4-2s}) & (0 \leq t \leq \frac{1}{4}s) \\ \gamma(t - \frac{1}{4}s) & (\frac{1}{4}s \leq t \leq \frac{1}{4} + \frac{1}{4}s) \\ \gamma(\frac{1}{1+s}t) & (\frac{1}{4} + \frac{1}{4}s \leq t \leq \frac{1}{2} + \frac{1}{2}s) \\ \gamma(\frac{2}{2-s}t - \frac{3s}{4-2s}) & (\frac{1}{2} + \frac{1}{2}s \leq t \leq 1), \end{cases}$$

the following shows the commutativity of (VII);

$$\begin{aligned} & H(\text{Comp})H(\Delta^! \otimes 1)(u \otimes x) \\ &= H(\xi \text{Comp} \circ T)H(\Delta^! \otimes 1)(u \otimes x) \\ &= H(T)H(\xi \text{Comp})(j^*(\rho \otimes \rho)\Delta^!(u) \cdot x) \\ &= H(T)\left((\xi \text{Comp})^* j^*(\rho \otimes \rho)\Delta^!(u) \cdot (\xi \text{Comp})^*(x)\right) \\ &= H(T)\left((ev_{1,0}, ev_{1,\frac{1}{2}})^*(\rho \otimes \rho)\Delta^!(u) \cdot (\xi \text{Comp})^*(x)\right) \\ &= H(T)H(\Delta^! \otimes 1)(u \otimes (\xi \text{Comp})^*(x)) \\ &= H(T)H(\Delta^! \otimes 1)(1 \otimes (\xi \text{Comp})^*(x)). \end{aligned}$$

for $u \otimes x$ in $H^*(\mathbb{B} \otimes_{A^{\otimes 2}} A_{\text{PL}}(LM))$. We conclude that

$$\begin{aligned} & Dlp \circ Dlcop \\ &= H(\Delta^! \otimes 1)\Phi'(1 \otimes \text{EM}_2'')^{-1}\text{EM}_1''H(T_{(123)})H(\text{inc})H(\text{Comp} \times 1) \\ &= (-1)^d H((1 \otimes \Delta^!) \otimes 1)H(\Delta^! \otimes (1 \otimes 1) \otimes 1)(1 \otimes (\text{EM}_5')^{-1})\text{EM}_5^{-1}H(\text{inc})H(\text{Comp} \times 1) \\ &= (-1)^d (1 \otimes Dlcop)(Dlp \otimes 1). \end{aligned}$$

A similar argument shows that the second equation $Dlp \circ Dlcop = (-1)^d(Dlcop \otimes 1)(1 \otimes Dlp)$. Consider the following three cubes:

$$(5.5) \quad \begin{array}{ccccc} & & LM \times_M LM \times_M LM & \xrightarrow{1 \times \text{Comp} \circ T} & LM \times_M LM \\ & \swarrow ev_0 & \downarrow (1 \times \text{Comp} \circ T)T_{(321)} & \swarrow (ev_2, \frac{1}{2}, ev_{2,0}) & \downarrow \xi' \text{Comp} \\ M & \xrightarrow{\Delta} & M \times 2 & \xrightarrow{\text{Comp}} & LM \\ \downarrow = & & \downarrow = & & \downarrow j \\ M & \xrightarrow{\Delta} & M \times 2 & \xrightarrow{\text{Comp}} & LM \end{array}$$

$$(5.6) \quad \begin{array}{ccccc} & & LM \times_M LM \times_M LM & \xrightarrow{\text{inc}} & (LM \times_M LM) \times LM \\ & \swarrow ev_0 & \downarrow \text{inc} & \swarrow ev_0 \times ev_0 & \downarrow \text{inc} \\ M & \xrightarrow{\Delta} & M \times 2 & \xrightarrow{\Delta \times 1} & LM \times 3 \\ \downarrow \Delta & & \downarrow \Delta \times 1 & & \downarrow \text{inc} \\ M \times M & \xrightarrow{1 \times \Delta} & M \times 3 & \xrightarrow{ev_0^{\times 3}} & LM \times 3 \end{array}$$

$$(5.7) \quad \begin{array}{ccccccc} & & LM \times_M LM \times_M LM & \xrightarrow{\text{inc}} & (LM \times_M LM) \times LM & \xrightarrow{\text{Comp} \times 1} & LM \times 2 \\ & \swarrow ev_0 & \downarrow T_{(123)} & \swarrow ev_0 \times ev_0 & \downarrow \Delta \times 1 & \swarrow j \times 1 & \downarrow R_{\frac{1}{2}} \times 1 \\ M & \xrightarrow{\Delta} & M \times 2 & \xrightarrow{\Delta \times 1} & M \times 3 & \xrightarrow{j \times 1} & LM \times 2 \\ \downarrow = & & \downarrow = & & \downarrow = & & \downarrow = \\ M & \xrightarrow{\Delta} & M \times 2 & \xrightarrow{1 \times \text{Comp} \circ T} & LM \times_M LM & \xrightarrow{\text{inc} \circ T} & LM \times 2 \\ & \swarrow ev_0 & \downarrow (ev_2, \frac{1}{2}, ev_{2,0}) & \swarrow (ev_2, \frac{1}{2}, ev_{2,0}) & \downarrow 1 \times \Delta & \swarrow (ev_{\frac{1}{2}}, ev_0) \times ev_0 & \downarrow \\ M & \xrightarrow{\Delta} & M \times 2 & \xrightarrow{1 \times \Delta} & M \times 3 & \xrightarrow{(ev_{\frac{1}{2}}, ev_0) \times ev_0} & LM \times 2 \end{array}$$

The map $\xi' : LM \rightarrow LM$ is given by

$$\xi'(\gamma)(t) = \begin{cases} \gamma(\frac{1}{2}t + \frac{3}{4}) & (0 \leq t \leq \frac{1}{2}) \\ \gamma(2t - 1) & (\frac{1}{2} \leq t \leq \frac{3}{4}) \\ \gamma(t - \frac{1}{4}) & (\frac{3}{4} \leq t \leq 1). \end{cases}$$

We denote by $R_{\frac{1}{2}}$ the rotation of loops by $\frac{1}{2}$, that is,

$$R_{\frac{1}{2}}(\gamma)(t) = \begin{cases} \gamma(t + \frac{1}{2}) & (0 \leq t \leq \frac{1}{2}) \\ \gamma(t - \frac{1}{2}) & (\frac{1}{2} \leq t \leq 1). \end{cases}$$

Denote by

$$\text{EM}''_3 : \mathbb{B} \otimes_{A \otimes 2} A_{\text{PL}}(LM \times_M LM) \longrightarrow A_{\text{PL}}(LM \times_M LM \times_M LM),$$

$$\text{EM}''_4 : (A \otimes \mathbb{B}) \otimes_{A \otimes 3} A_{\text{PL}}(LM \times LM) \longrightarrow A_{\text{PL}}(LM \times_M LM)$$

the Eilenberg-Moore maps of the top square of the diagram (5.5) and the bottom and right-hand side square of (5.7), respectively. Consider the following diagram:

(5.8)

$$\begin{array}{ccccccc}
& & & & (1 \otimes Dlp) & & \\
H^*(LM \times LM) & \xrightarrow{H(1 \times \text{Comp})} & H^*(LM \times (LM^{\times M^2})) & \xrightarrow[\cong]{(EM_1)^{-1}} & H^*((A \otimes \mathbb{B}) \otimes_{A \otimes 3} A_{PL}(LM^{\times 3})) & \xrightarrow{H((1 \otimes \Delta^!) \otimes 1)} & H^*(LM^{\times 3}) \\
\downarrow H(\text{inc}) & (I') & \downarrow H(\text{inc}) & (II') & \downarrow \Delta^* \otimes_{(\Delta \times 1)^*} \text{inc}^* & (III') & \downarrow H(\text{inc} \times 1) \\
H^*(LM \times_M LM) & \xrightarrow{H(1 \times_M \text{Comp} \circ T)} & H^*(LM^{\times M^3}) & \xrightarrow[\cong]{(EM_5)^{-1}} & H^*(\mathbb{B} \otimes_{A \otimes 2} A_{PL}((LM^{\times M^2}) \times LM)) & \xrightarrow{H(\Delta^! \otimes 1)} & H^*((LM^{\times M^2}) \times LM) \\
\downarrow \cong & (EM')^{-1} & \downarrow \cong & H(T_{(123)}) = H(T_{(321)})^{-1} & \downarrow \cong & (1 \otimes EM'_1)^{-1} & (VI') & \downarrow (EM'_1)^{-1} \cong \\
& (IV') & H^*(LM^{\times M^3}) & (V') & H^*(\mathbb{B} \otimes_{A \otimes 2} ((\mathbb{B} \otimes A) \otimes_{A \otimes 3} A_{PL}(LM^{\times 2}))) & \xrightarrow{H(\Delta^! \otimes (1 \otimes 1) \otimes 1)} & H^*((\mathbb{B} \otimes A) \otimes_{A \otimes 3} A_{PL}(LM^{\times 2})) \\
& & \downarrow \cong & (EM'_3)^{-1} & \downarrow \cong & \Psi & \\
H^*(\mathbb{B} \otimes_{A \otimes 2} A_{PL}(LM)) & \xrightarrow{1 \otimes (\xi' \text{Comp})^*} & H^*(\mathbb{B} \otimes_{A \otimes 2} A_{PL}(LM^{\times M^2})) & \xrightarrow[\cong]{(1 \otimes EM'_4)^{-1}} & H^*(\mathbb{B} \otimes_{A \otimes 2} ((A \otimes \mathbb{B}) \otimes_{A \otimes 3} A_{PL}(LM^{\times 2}))) & (IX') & H((\Delta^! \otimes 1) \otimes 1) \\
\downarrow H(\Delta^! \otimes 1) & (VII') & \downarrow H(T)H(\Delta^! \otimes 1) & (VIII') & \downarrow \Psi' & & \downarrow \\
H^*(LM) & \xrightarrow{H(\text{Comp})} & H^*(LM^{\times M^2}) & \xrightarrow[\cong]{(EM)^{-1}} & H^*(\mathbb{B} \otimes_{A \otimes 2} A_{PL}(LM^{\times 2})) & \xrightarrow{(-1)^d H(\Delta^! \otimes 1)} & H^*(LM \times LM). \\
& & & Dlp & & &
\end{array}$$

We defined the map Ψ and Ψ' for

$$\begin{aligned}\Psi(v \otimes (a \otimes u) \otimes x) &= (-1)^{|u|(|a|+|v|)+|a||v|} u \otimes (av \otimes 1) \otimes (R_{\frac{1}{2}} \times 1)^*(x), \\ \Psi'(v \otimes (a \otimes u) \otimes x) &= \sum (-1)^{|v_2||a|+(|v_1|+|a|)(|v_2|+|u|)} v_2 u \otimes \left(ev_{1, \frac{1}{2}}^* \rho(v_1 a) \cdot x \right),\end{aligned}$$

respectively. We also see that Ψ is an isomorphism with the inverse map Ψ^{-1} given by

$$\Psi^{-1}(v \otimes (u \otimes a) \otimes x) = (-1)^{|v|(|a|+|u|)} u \otimes (1 \otimes av) \otimes (R_{\frac{1}{2}} \times 1)^*(x).$$

Similar for the argument of the proof the commutativity of the diagram (5.4), we see that the diagram (5.8) commutes. We only show the commutativity for the diagrams (I') and (IX'). Since the composite $\text{Comp} \circ T$ is homotopic to Comp , the diagrams (I') commutes. It is readily seen that the equality

$$H(R_{\frac{1}{2}} \times 1)H((\Delta^! \otimes 1) \otimes 1)H(\Delta^! \otimes (1 \otimes 1) \otimes 1)\Psi = (-1)^d H(\Delta^! \otimes 1)\Psi'$$

holds by a straightforward calculation. Since the map $R_{\frac{1}{2}}$ is homotopic to the identity map, we therefore conclude that

$$Dlp \circ Dlcop = (-1)^d (Dlcop \otimes 1)(1 \otimes Dlp).$$

□

6. A TRIVIALITY OR NON-TRIVIALITY OF THE LOOP PRODUCT AND THE LOOP COPRODUCT.

We first introduce a semifree resolution of a minimal Sullivan model $(\Lambda V, d)$ for M as a $\Lambda V \otimes \Lambda V$ -module. Consider the commutative graded algebra $\Lambda V \otimes \Lambda V \otimes \Lambda(sV)$ with the differential D given by

$$\begin{aligned}D(v \otimes 1 \otimes 1) &= d(v) \otimes 1 \otimes 1, D(1 \otimes v \otimes 1) = 1 \otimes d(v) \otimes 1, \\ D(1 \otimes 1 \otimes sv) &= (-v \otimes 1 + 1 \otimes v) \otimes 1 - \sum_{i=1}^{\infty} \frac{(sD)^i}{i!} (v \otimes 1 \otimes 1),\end{aligned}$$

where sV is the suspension of V , that is $(sV)^n = V^{n+1}$, and s is the unique derivation of the algebra $\Lambda V \otimes \Lambda V \otimes \Lambda(sV)$ defined by

$$s(v \otimes 1 \otimes 1) = 1 \otimes 1 \otimes sv = s(1 \otimes v \otimes 1), \quad s(1 \otimes 1 \otimes sv) = 0.$$

Then, by [7, §15, Example1], $(\Lambda V \otimes \Lambda V \otimes \Lambda(sV), D)$ is a Sullivan model for M^I and

$$\bar{\varepsilon} := \mu \cdot \varepsilon : \Lambda V \otimes \Lambda V \otimes \Lambda(sV) \rightarrow \Lambda V$$

is a semifree resolution of ΛV as a $\Lambda V \otimes \Lambda V$ -module. Here, μ is the product of ΛV and $\varepsilon : \Lambda V \rightarrow \mathbb{Q}$ is the canonical augmentation. Moreover, the commutative differential graded algebra

$$(\Lambda V \otimes \Lambda(sV), \bar{d}) \cong (\Lambda V, d) \otimes_{\Lambda V \otimes \Lambda V} (\Lambda V \otimes \Lambda V \otimes \Lambda(sV), D)$$

is a Sullivan model for the free loop space LM , where the differential \bar{d} is defined as $\bar{d}(v) = d(v)$, $\bar{d}(sv) = -sd(v)$. For simplicity, we put

$$\mathcal{M}_{M^I} = (\Lambda V \otimes \Lambda V \otimes \Lambda(sV), D), \quad \mathcal{M}_{LM} = (\Lambda V \otimes \Lambda(sV), \bar{d}).$$

We next consider a model for the dual loop product and coproduct. The following is a rational coefficient version of the torsion functor description of [12]. If M is a

Poincaré duality space, models for the products are introduced by [3], [9] and [10]. By the following commutative diagram,

$$\begin{array}{ccccc}
 & LM \times_M LM & \xrightarrow{pr_2} & LM & \\
 \swarrow pr_1 & \downarrow \text{Comp} & & \swarrow ev_0 & \downarrow inc \\
 LM & \xrightarrow{ev_0} & M & \xleftarrow{ev_0} & LM \\
 \downarrow inc & \downarrow \theta & \downarrow \zeta_1 & \downarrow = & \downarrow inc \\
 M^I & \xleftarrow{\zeta_2} & LM & \xrightarrow{(ev_1, ev_0)} & M \times M & \xleftarrow{(ev_0, ev_1)} & M^I \\
 \uparrow c & \uparrow ev_0 & \uparrow inc & \uparrow = & \uparrow (ev_0, ev_1) & & \\
 M & \xrightarrow{\Delta} & M \times M & & & &
 \end{array}$$

the morphism

$$\begin{array}{ccc}
 \mathcal{M}_{LM} \cong \Lambda V \otimes_{\Lambda V \otimes 2} \mathcal{M}_{M^I} & \xleftarrow[\simeq]{\bar{\varepsilon} \otimes 1} & \mathcal{M}_{M^I} \otimes_{\Lambda V \otimes 2} \mathcal{M}_{M^I} \\
 & & \downarrow (\mu \otimes 1) \otimes_{\mu} (\mu \otimes 1) \\
 \Lambda V \otimes_{\Lambda V \otimes 2} \mathcal{M}_{LM}^{\otimes 2} & \xleftarrow[\cong]{} & \mathcal{M}_{LM} \otimes_{\Lambda V} \mathcal{M}_{LM}
 \end{array}$$

induces the map $H(\text{Comp}) = H((\mu \otimes 1) \otimes_{\mu} (\mu \otimes 1))H(\bar{\varepsilon} \otimes 1)^{-1}$ in homology. Here, $c : M \rightarrow LM$ is the map which sends x to the constant path at x . The maps θ , ζ_1 and ζ_2 are given as follows:

$$\theta(\gamma)(t) = \begin{cases} \gamma(2t) & (0 \leq t \leq \frac{1}{2}) \\ \gamma(1) & (\frac{1}{2} \leq t \leq 1) \end{cases}$$

and $\zeta_1(\gamma)(t) = \gamma(\frac{1}{2}t)$, $\zeta_2(\gamma)(t) = \gamma(\frac{1}{2}t + \frac{1}{2})$ for $\gamma \in LM$. Hence, the dual loop product is induced by the following composite in homology

$$\begin{array}{ccc}
 (6.1) \quad \mathcal{M}_{LM} & \xrightarrow[\cong]{} & \Lambda V \otimes_{\Lambda V \otimes 2} \mathcal{M}_{M^I} \xleftarrow[\simeq]{\bar{\varepsilon} \otimes 1} \mathcal{M}_{M^I} \otimes_{\Lambda V \otimes 2} \mathcal{M}_{M^I} \\
 & & \downarrow (\mu \otimes 1) \otimes_{\mu} (\mu \otimes 1) \\
 & & \mathcal{M}_{LM} \otimes_{\Lambda V} \mathcal{M}_{LM} \\
 & & \downarrow \cong \\
 \mathcal{M}_{LM}^{\otimes 2} & \xleftarrow[\simeq]{\Delta^! \otimes 1 \otimes 1} \mathcal{M}_{M^I} \otimes_{\Lambda V \otimes 2} \mathcal{M}_{LM}^{\otimes 2} & \xrightarrow[\simeq]{\bar{\varepsilon} \otimes 1} \Lambda V \otimes_{\Lambda V \otimes 2} \mathcal{M}_{LM}^{\otimes 2}.
 \end{array}$$

Consider the two commutative diagrams,

$$\begin{array}{ccccc}
 & M \times M & & & \\
 & \downarrow j & & & \\
 & LM & \xrightarrow{\zeta_1} & M^I & \\
 \swarrow \zeta_2 & \downarrow j & \swarrow (ev_1, ev_0) & \swarrow (ev_0, ev_1) & \downarrow (ev_0, ev_1) \\
 M^I & \xrightarrow{(ev_1, ev_0)} & M \times M & \xrightarrow{(ev_0, ev_1)} & M^I \\
 \downarrow (ev_1, ev_0) & \downarrow = & \downarrow = & \downarrow = & \\
 M \times M & \xrightarrow{=} & M \times M & \xrightarrow{=} & M \times M
 \end{array}$$

$$(6.3) \quad \begin{array}{ccccc} & LM \times_M LM & \xrightarrow{\text{Comp}} & LM & \\ \swarrow ev_0 & \downarrow \text{inc} & & \swarrow j & \\ M & \xrightarrow{\Delta} M \times M & & & \downarrow (\zeta_1, \zeta_2) \\ \downarrow \Delta & LM \times LM & \xrightarrow{\text{inc}} & M^I \times M^I & \\ (ev_0, ev_0) \swarrow & \downarrow \Delta' & & \swarrow (ev_0, ev_1)^{\times 2} & \\ M \times M & \xrightarrow{\Delta \times \Delta} M^{\times 2} \times M^{\times 2} & & & \end{array}$$

where Δ' is a map which sends (x, y) to (x, y, y, x) . By the diagram (6.2), we see that the inclusion $\Lambda V \otimes \Lambda V \hookrightarrow \mathcal{M}_{M^I} \otimes_{\Lambda V^{\otimes 2}} \mathcal{M}_{M^I}$ is a model for j . Since the quotient map

$$\bar{\zeta} : \mathcal{M}_{M^I} \otimes \mathcal{M}_{M^I} \longrightarrow \mathcal{M}_{M^I} \otimes_{\Lambda V^{\otimes 2}} \mathcal{M}_{M^I}$$

is a model for (ζ_1, ζ_2) , by the diagram (6.3), the following composite is a model for $\text{inc} : LM \times_M LM \rightarrow LM \times LM$;

$$\mathcal{M}_{LM}^{\otimes 2} \cong \Lambda V^{\otimes 2} \otimes_{\Lambda V^{\otimes 4}} \mathcal{M}_{M^I}^{\otimes 2} \xrightarrow{\mu \otimes_{\mu'} \bar{\zeta}} \Lambda V \otimes_{\Lambda V^{\otimes 2}} (\mathcal{M}_{M^I} \otimes_{\Lambda V^{\otimes 2}} \mathcal{M}_{M^I}),$$

where μ' is a model for Δ' , that is, $\mu'(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = (-1)^{|v_4|(|v_2|+|v_3|)} v_1 v_4 \otimes v_2 v_3$. We thus see that the dual loop coproduct is induced by the composite in homology;

$$(6.4) \quad \begin{array}{ccc} \mathcal{M}_{LM}^{\otimes 2} \cong \Lambda V^{\otimes 2} \otimes_{\Lambda V^{\otimes 4}} \mathcal{M}_{M^I}^{\otimes 2} & \xrightarrow{\mu \otimes_{\mu'} \bar{\zeta}} & \Lambda V \otimes_{\Lambda V^{\otimes 2}} (\mathcal{M}_{M^I} \otimes_{\Lambda V^{\otimes 2}} \mathcal{M}_{M^I}) \\ & & \uparrow \simeq \bar{\varepsilon} \otimes 1 \\ & & \mathcal{M}_{M^I} \otimes_{\Lambda V^{\otimes 2}} (\mathcal{M}_{M^I} \otimes_{\Lambda V^{\otimes 2}} \mathcal{M}_{M^I}) \\ & & \downarrow \Delta' \otimes 1 \\ \mathcal{M}_{LM} & \xleftarrow[\simeq]{\bar{\varepsilon} \otimes 1} & \mathcal{M}_{M^I} \otimes_{\Lambda V^{\otimes 2}} \mathcal{M}_{M^I} \end{array}$$

The following is the result of Félix, Halperin and Thomas related to rational Gorenstein spaces.

Theorem 6.1. [6, Proposition 3.4, Proposition 5.1] *Let X be simply-connected space and assume that the rational homotopy group $\pi_*(X) \otimes \mathbb{Q}$ is finite dimension. Then, X is a \mathbb{Q} -Gorenstein space with formal dimension*

$$\sum_{|x_i|:\text{odd}} |x_i| - \sum_{|x_i|:\text{even}} (|x_i| - 1),$$

where x_i is a basis of $\pi_*(X) \otimes \mathbb{Q}$.

Let $(\Lambda V, d)$ be a minimal Sullivan model for a simply-connected space M . Since V is isomorphic to $\text{Hom}_{\mathbb{Z}}(\pi_*(X), \mathbb{Q})$ ([7, Lemma 13.11]), if V is finite dimensional, then M is a Gorenstein space. We now put

$$\text{fdim } M = \sum_{|x_i|:\text{odd}} |x_i| - \sum_{|x_i|:\text{even}} (|x_i| - 1).$$

Before proving Proposition 1.3, we give the following lemma.

Lemma 6.2. *For any odd degree elements $x_1, x_2 \cdots x_k$ in V ,*

$$\left(\prod_{i=1}^k (-x_i \otimes 1 + 1 \otimes x_i) \right) (-x_1 x_2 \cdots x_k \otimes 1 + 1 \otimes x_1 x_2 \cdots x_k) = 0$$

in $\Lambda V \otimes \Lambda V$.

Proof. The proof is induction on k . For $k = 1$, it is easily seen that $(-x_1 \otimes 1 + 1 \otimes x_1)^2 = 0$. Assume that the equation

$$\left(\prod_{i=1}^{k-1} (-x_i \otimes 1 + 1 \otimes x_i) \right) (-x_1 x_2 \cdots x_{k-1} \otimes 1 + 1 \otimes x_1 x_2 \cdots x_{k-1}) = 0$$

hold. Then,

$$\begin{aligned} & (-x_k \otimes 1 + 1 \otimes x_k) (-x_1 x_2 \cdots x_k \otimes 1 + 1 \otimes x_1 x_2 \cdots x_k) \\ &= (-x_k \otimes 1 + 1 \otimes x_k) \left((-x_1 x_2 \cdots x_{k-1} \otimes 1 + 1 \otimes x_1 x_2 \cdots x_{k-1}) (x_k \otimes 1 + 1 \otimes x_k) \right. \\ &\quad \left. - (-1)^{k-1} x_k \otimes x_1 x_2 \cdots x_{k-1} + x_1 x_2 \cdots x_{k-1} \otimes x_k \right) \\ &= (-1)^k (x_k \otimes x_k) (-x_1 x_2 \cdots x_{k-1} \otimes 1 + 1 \otimes x_1 x_2 \cdots x_{k-1}). \end{aligned}$$

Hence, induction hypothesis shows the assertion. \square

Proof of Proposition 1.2. (1) Let x_1, x_2, \dots, x_n be a homogeneous basis of $V = V^{\text{odd}}$. We see that a $\Lambda V^{\otimes 2}$ -module map $\Delta^! : \mathcal{M}_{M^I} \rightarrow \Lambda V^{\otimes 2}$ defined by

$$\Delta^!(1) = \prod_{i=1}^n (-x_i \otimes 1 + 1 \otimes x_i), \quad \Delta^!(sx_{i_1} \cdots sx_{i_k}) = 0$$

is a generator of $\text{Ext}_{\Lambda V^{\otimes 2}}^{\text{fdim } M}(\Lambda V, \Lambda V^{\otimes 2})$. We now check that the map $\Delta^!$ is a cycle but it is not boundary. For any x_i , we may write $dx_i = \sum \lambda x_{i_1} \cdots x_{i_k}$ for some $\lambda \in \mathbb{Q}$. Then, by Lemma 6.2,

$$d\Delta^!(1) = d\left(\prod_{i=1}^n (-x_i \otimes 1 + 1 \otimes x_i)\right) = 0$$

Hence, $(d\Delta^! - (-1)^d \Delta^! D)(1) = 0$. The equality $(-x_i \otimes 1 + 1 \otimes x_i)^2 = 0$ enables us to obtain that $(d\Delta^! - (-1)^d \Delta^! D)(sx_i) = 0$, and similarly, we have $(d\Delta^! - (-1)^d \Delta^! D)(sx_{i_1} \cdots sx_{i_k}) = 0$. Hence $\Delta^!$ is a cycle in $\text{Hom}_{\Delta V^{\otimes 2}}^*(\mathcal{M}_{M^I}, \Lambda V^{\otimes 2})$. Any $\Lambda V^{\otimes 2}$ -module map $\mathcal{M}_{M^I} \rightarrow \Lambda V^{\otimes 2}$ with degree $\text{fdim } M - 1$ send 1 to 0 by degree reason, the map $\Delta^!$ is not a boundary. Since $\Delta^!(1)$ is a non-trivial cycle, the dual loop product is non-trivial. The equality $\mu\Delta^! = 0$ implies that the dual loop coproduct is trivial.

(2) We first note that if V is generated by even degree basis, the differential d is zero by degree reason. Let y_1, y_2, \dots, y_m be a basis of $V = V^{\text{even}}$. Then, the $\Lambda V^{\otimes 2}$ -module map $\Delta^! : \mathcal{M}_{M^I} \rightarrow \Lambda V^{\otimes 2}$ defined by $\Delta^!(1) = 0$ and

$$\Delta^!(sy_{j_1} sy_{j_2} \cdots sy_{j_l}) = \begin{cases} 1 & (\{j_1, j_2, \dots, j_l\} = \{1, 2, \dots, m\}) \\ 0 & (\text{otherwise}). \end{cases}$$

is a generator of $\text{Ext}_{\Lambda V^{\otimes 2}}^{\text{fdim } M}(\Lambda V, \Lambda V^{\otimes 2})$. Indeed, a straightforward calculation shows that the map $\Delta^!$ is compatible with the differentials. Also, for any $\Lambda V^{\otimes 2}$ -module map $\psi : \mathcal{M}_{M^I} \rightarrow \Lambda V^{\otimes 2}$ of degree $\text{fdim } M - 1$, the following equation shows that $\Delta^!$ is not a boundary:

$$\begin{aligned} & (-1)^{\text{fdim } M - 1} \psi D((sy_1 \cdots sy_m)) \\ &= (-1)^{\text{fdim } M - 1} \sum_{i=1}^m \pm \psi((-y_i \otimes 1 + 1 \otimes y_i) \otimes sy_1 \cdots sy_{i-1} sy_{i+1} \cdots sy_m) \\ &= (-1)^{\text{fdim } M - 1} \sum_{i=1}^m \pm (-y_i \otimes 1 + 1 \otimes y_i) \psi(sy_1 \cdots sy_{i-1} sy_{i+1} \cdots sy_m) \neq 1, \end{aligned}$$

where \pm is the Kuszul sign convention. By (6.1), we have that

$$\begin{aligned} & H(\bar{\varepsilon} \otimes 1)^{-1} H(\text{Comp})(v_1 v_2 \cdots v_l \otimes s w_1 s w_2 \cdots s w_k) \\ &= \prod_{i=1}^k (v_1 \cdots v_l \otimes 1 \otimes 1) \otimes \left((1 \otimes 1) \otimes (1 \otimes s w_i) - (1 \otimes s w_i) \otimes (1 \otimes 1) \right). \end{aligned}$$

for any $v_1 v_2 \cdots v_l \otimes s w_1 \cdots s w_k$ in \mathcal{M}_{LM} . Therefore, $\Delta^!(1) = 0$ implies that the dual loop product is trivial. Also the following equations show that the dual loop coproduct is non-trivial:

$$\begin{aligned} & Dlcop((1 \otimes s y_1 s y_2 \cdots s y_m) \otimes (1 \otimes 1)) \\ &= H(\bar{\varepsilon} \otimes 1) H(\Delta^! \otimes 1) \left(\prod_{i=1}^m (-(1 \otimes 1 \otimes s y_i) \otimes (1 \otimes 1 \otimes 1) \otimes (1 \otimes 1 \otimes 1)) \right. \\ & \quad \left. (1 \otimes 1 \otimes 1) \otimes (1 \otimes 1 \otimes s y_i) \otimes (1 \otimes 1 \otimes 1) \right) \\ &= H(\bar{\varepsilon} \otimes 1) ((-1)^m (1 \otimes 1 \otimes 1) \otimes (1 \otimes 1 \otimes 1)) = (-1)^m (1 \otimes 1) \neq 0. \end{aligned}$$

This completes the proof. \square

We next consider spaces in which a minimal Sullivan model of the spaces are pure and first recall the definition of pure Sullivan algebras.

Definition 6.3. [7, §32 (a)] A minimal Sullivan model is *pure* if V is finite dimensional, $d(V^{\text{odd}}) \subset \Lambda V^{\text{even}}$ and $d(V^{\text{even}}) = 0$.

Let $(\Lambda V, d)$ be a pure minimal Sullivan model with $V^{\text{odd}} \neq \{0\}$, $V^{\text{even}} \neq \{0\}$ and x_1, \dots, x_n is a basis of V^{odd} and y_1, \dots, y_m is a basis of V^{even} . Then, we may write

$$D(s x_r) = (-x_r \otimes 1 + 1 \otimes x_r) \otimes 1 - \sum_{i=1}^m f_i^r \otimes s y_i$$

in \mathcal{M}_{M^I} for some $f_i^r \in \Lambda V^{\otimes 2}$. For a subset $J = \{j_1 < j_2 < \cdots < j_k\}$ of $\{1, 2, \dots, m\}$, we put

$$s y_J = s y_{j_1} s y_{j_2} \cdots s y_{j_k}$$

for simplicity. Especially, if J is the empty set ϕ , put $s y_\phi = 1$. We now define a $\Lambda V^{\otimes 2}$ -module map $\Delta^! : \mathcal{M}_{M^I} \rightarrow \Lambda V^{\otimes 2}$ as follows: for any generator of $\Lambda(sV)$, if a generator is of the form $s y_{J^c}$ for some $J \subset \{1, 2, \dots, m\}$,

$$\Delta^!(s y_{J^c}) = \sum_{i_1=1}^n \sum_{\substack{i_2=1, \\ i_2 \neq i_1}}^n \cdots \sum_{\substack{i_k=1, \\ i_k \neq i_1, \dots, i_{k-1}}}^n (-1)^{\varepsilon(J, i_1, \dots, i_k)} f_{j_1}^{i_1} \cdots f_{j_k}^{i_k} \left(\prod_{\substack{i=1, \\ i \neq i_1, \dots, i_k}}^n (-x_i \otimes 1 + 1 \otimes x_i) \right)$$

and $\Delta^!$ sends the others to 0. Here, J^c is the complementary subset of J ,

$$\varepsilon(J, i_1, \dots, i_k) = \sum_{r=1}^k (i_r + j_r + r - 1) + k \dim M + s(\sigma),$$

σ is a k -permutation which satisfies $i_{\sigma(k)} < i_{\sigma(k-1)} < \cdots < i_{\sigma(1)}$. If σ is an even permutation, we put $s(\sigma) = 0$ and if σ is an odd permutation, put $s(\sigma) = 1$.

Lemma 6.4. *The map $\Delta^!$ is a generator of $\text{Ext}_{\Lambda V^{\otimes 2}}^{\text{fdim } M}(\Lambda V, \Lambda V^{\otimes 2})$.*

Proof. It is only enough to check that the map $\Delta^!$ is a cycle and not a boundary in $\text{Hom}_{\Lambda V^{\otimes 2}}^*(\mathcal{M}_{M^I}, \Lambda V^{\otimes 2})$. Let $\{i_1, i_2, \dots, i_k\}$ be a subset of $\{1, 2, \dots, n\}$ and σ

is a k -permutation such that $i_{\sigma(k)} < i_{\sigma(k-1)} < \cdots < i_{\sigma(1)}$. Then,

$$\begin{aligned} & d\left(\prod_{\substack{i=1, \\ i \neq i_1, \dots, i_k}}^n (-x_i \otimes 1 + 1 \otimes x_i)\right) \\ &= \sum_{r=0}^k \sum_{p=i_{\sigma(k-r+1)}+1}^{i_{\sigma(k-r)}-1} (-1)^{p-1-r} (-dx_p \otimes 1 + 1 \otimes dx_p) \prod_{\substack{i=1, \\ i \neq i_1, \dots, i_k, p}}^n (-x_i \otimes 1 + 1 \otimes x_i), \end{aligned}$$

where we put $\sigma(k+1) = 0$ and $\sigma(0) = n+1$ for convenient. Since $-dx_p \otimes 1 + 1 \otimes dx_p = \sum_{j=1}^m f_j^p (-y_j \otimes 1 + 1 \otimes y_j)$,

$$\begin{aligned} d\Delta^!(sy_{J^c}) &= \sum_{j=1}^m \sum_{r=0}^k \sum_{p=i_{\sigma(k-r+1)}+1}^{i_{\sigma(k-r)}-1} \sum_{i_1=1}^n \sum_{\substack{i_2=1, \\ i_2 \neq i_1}}^n \cdots \sum_{\substack{i_k=1, \\ i_k \neq i_1, \dots, i_{k-1}}}^n (-1)^{\varepsilon_{(J, i_1, \dots, i_k)} + p-1-r} \\ &\quad \times (-y_j \otimes 1 + 1 \otimes y_j) f_j^p f_{j_1}^{i_1} \cdots f_{j_k}^{i_k} \prod_{\substack{i=1, \\ i \neq i_1, \dots, i_k, p}}^n (-x_i \otimes 1 + 1 \otimes x_i). \end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned} & \sum_{j \in J} \sum_{r=0}^k \sum_{p=i_{\sigma(k-r+1)}+1}^{i_{\sigma(k-r)}-1} \sum_{i_1=1}^n \sum_{\substack{i_2=1, \\ i_2 \neq i_1}}^n \cdots \sum_{\substack{i_k=1, \\ i_k \neq i_1, \dots, i_{k-1}}}^n (-1)^{\varepsilon_{(J, i_1, \dots, i_k)} + p-1-r} \\ & \quad \times (-y_j \otimes 1 + 1 \otimes y_j) f_j^p f_{j_1}^{i_1} \cdots f_{j_k}^{i_k} \prod_{\substack{i=1, \\ i \neq i_1, \dots, i_k, p}}^n (-x_i \otimes 1 + 1 \otimes x_i) = 0 \end{aligned}$$

and

$$\begin{aligned} & \sum_{j \in J^c} \sum_{r=0}^k \sum_{p=i_{\sigma(k-r+1)}+1}^{i_{\sigma(k-r)}-1} \sum_{i_1=1}^n \sum_{\substack{i_2=1, \\ i_2 \neq i_1}}^n \cdots \sum_{\substack{i_k=1, \\ i_k \neq i_1, \dots, i_{k-1}}}^n (-1)^{\varepsilon_{(J, i_1, \dots, i_k)} + p-1-r} \\ & \quad \times (-y_j \otimes 1 + 1 \otimes y_j) f_j^p f_{j_1}^{i_1} \cdots f_{j_k}^{i_k} \prod_{\substack{i=1, \\ i \neq i_1, \dots, i_k, p}}^n (-x_i \otimes 1 + 1 \otimes x_i) = (-1)^{\text{fdim } M} \Delta^! D(sy_{J^c}). \end{aligned}$$

Hence, the equation $(d\Delta^! - (-1)^{\text{fdim } M} \Delta^! D)(sy_{J^c}) = 0$ holds. Similarly, for a base of $\Lambda(sV)$ of the form $sx_q sy_{J^c}$, we see that $(d\Delta^! - (-1)^{\text{fdim } M} \Delta^! D)(sx_q sy_{J^c}) = 0$. Indeed,

$$\begin{aligned} & (d\Delta^! - (-1)^{\text{fdim } M} \Delta^! D)(sx_q sy_{J^c}) \\ &= (-1)^{\text{fdim } M+1} \Delta^! D(sx_q sy_{J^c}) \\ &= -(-x_q \otimes 1 + 1 \otimes x_q) \Delta^!(sy_{J^c}) + (-1)^{\text{fdim } M} \sum_{r=1}^k (-1)^{j_r-r} f_{j_r}^q \Delta^!(sy_{(J-\{j_r\})^c}). \end{aligned}$$

Since

$$\begin{aligned}
(6.5) \quad & \sum_{r=1}^k (-1)^{\text{fdim } M + j_r - r} f_{j_r}^q \Delta^! (sy_{(J - \{j_r\})^c}) \\
&= \sum_{r=1}^k \sum_{i_1=1}^n \cdots \sum_{\substack{i_{r-1}=1, \\ i_{r-1} \neq i_1, \dots, i_{r-2}}}^n \sum_{\substack{i_{r+1}=1, \\ i_{r+1} \neq i_1, \dots, i_{r-1}}}^n \cdots \sum_{\substack{i_k=1, \\ i_k \neq i_1, \dots, i_{k-1}}}^n \\
& \quad (-1)^{\text{fdim } M + j_r - r + \varepsilon_{(J - \{j_r\}, i_1, \dots, i_k)}} f_{j_r}^q f_{j_1}^{i_1} \cdots f_{j_k}^{i_k} \left(\prod_{\substack{i=1, \\ i \neq i_1, \dots, i_k}}^n (-x_i \otimes 1 + 1 \otimes x_i) \right),
\end{aligned}$$

we can decompose the right-hand side of (6.5) to the following two terms;

$$\begin{aligned}
(6.6) \quad & \sum_{r=1}^k \sum_{\substack{i_1=1, \\ i_1 \neq q}}^n \cdots \sum_{\substack{i_{r-1}=1, \\ i_{r-1} \neq q, i_1, \dots, i_{r-2}}}^n \sum_{\substack{i_{r+1}=1, \\ i_{r+1} \neq q, i_1, \dots, i_{r-1}}}^n \cdots \sum_{\substack{i_k=1, \\ i_k \neq q, i_1, \dots, i_{k-1}}}^n \\
& \quad (-1)^{\text{fdim } M + j_r - r + \varepsilon_{(J - \{j_r\}, i_1, \dots, i_k)}} f_{j_r}^q f_{j_1}^{i_1} \cdots f_{j_k}^{i_k} \left(\prod_{\substack{i=1, \\ i \neq i_1, \dots, i_k}}^n (-x_i \otimes 1 + 1 \otimes x_i) \right)
\end{aligned}$$

and the sum of the other terms. A straightforward calculation shows that the term (6.6) is equal to $(-x_q \otimes 1 + 1 \otimes x_q) \Delta^! (sy_{J^c})$ and the others is zero. Hence, $(d\Delta^! - (-1)^{\text{fdim } M} \Delta^! D)(sx_q sy_{J^c}) = 0$. It is readily seen that $(d\Delta^! - (-1)^{\text{fdim } M} \Delta^! D)(sx_1^{m_1} \cdots sx_n^{m_n} sy_{J^c}) = 0$ for some $m_i \geq 0$, it turns out that $\Delta^!$ is a cycle in $\text{Hom}_{\Lambda V^{\otimes 2}}^*(\mathcal{M}_{M^I}, \Lambda V^{\otimes 2})$. By definition of $\Delta^!$,

$$\Delta^! (sy_1 \cdots sy_m) = \prod_{i=1}^m (-x_i \otimes 1 + 1 \otimes x_i).$$

However, for any $\Lambda V^{\otimes 2}$ -module map ψ with degree $\text{fdim } M - 1$, we see that $(d\psi - (-1)^{\text{fdim } M - 1} \psi D)(sy_1 \cdots sy_m)$ is not in $\Lambda^{\geq 1} V^{\text{odd}}$. It implies that $\Delta^!$ is not a boundary, and hence we have the assertion. \square

Proof of Proposition 1.4. (1) If the differential d is zero, $f_i^r = 0$ for any r and i . It turns out that, by the formula of Lemma 6.4, $\Delta^!(1) = 0$ and $\mu\Delta^! = 0$. Similarly argument of the proof of Proposition 1.3 (2), the equality $\Delta^!(1) = 0$ implies that the dual loop product is trivial. We also see that the dual loop coproduct is trivial by the equation $\mu\Delta^! = 0$.

(2) By Lemma 6.4, if $\dim V^{\text{odd}} > \dim V^{\text{even}}$, then $\mu\Delta^! = 0$. Therefore, we see that the dual loop coproduct is trivial. \square

Example 6.5. Let $M = ES^1 \times_{S^1} \mathbb{C}P^2$ be the Borel construction associated to the action

$$S^1 \times \mathbb{C}P^2 \longrightarrow \mathbb{C}P^2, \quad t \cdot (x, y, z) = (tx, y, z).$$

We see that the space M is a \mathbb{Q} -Gorenstein space of formal dimension 3 ([6, Theorem 4.3]). By [8, Example 7.41], a commutative differential graded algebra $(\Lambda V, d) = (\Lambda(x_2, u_2, w_5), d)$ with $|x_2| = |u_2| = 2$, $|w_5| = 5$ and $dx_2 = du_2 = 0$, $dw_5 = u_2^3 + x_2 u_2^2$ is a minimal pure Sullivan model for $ES^1 \times_{S^1} \mathbb{C}P^2$. A straightforward computation

shows that

$$\begin{aligned} D(sw_5) &= (-w_5 \otimes 1 + 1 \otimes w_5) \otimes 1 - f \otimes su_2 - g \otimes sx_2, \\ f &= u_2^2 \otimes 1 + u_2 \otimes u_2 + 1 \otimes u_2^2 + \frac{1}{3}u_2 \otimes x_2 + \frac{1}{3}x_2 \otimes u_2 + \frac{2}{3}u_2x_2 \otimes 1 + \frac{2}{3} \otimes u_2x_2, \\ g &= \frac{1}{3}u_2^2 \otimes 1 + \frac{1}{3}u_2 \otimes u_2 + \frac{1}{3} \otimes u_2^2 \end{aligned}$$

in \mathcal{M}_{MI} . Hence, by Lemma 6.4, the $\Lambda V^{\otimes 2}$ -module map $\Delta^!$ satisfies that

$$\Delta^!(sx_2su_2) = -w_5 \otimes 1 + 1 \otimes w_5, \quad \Delta^!(sx_2) = f, \quad \Delta^!(su_2) = -g, \quad \Delta^!(1) = 0.$$

It is easily seen that the dual loop coproduct is non-trivial. Indeed, $1 \otimes su_2$ is a non-zero element in $H^*(\mathcal{M}_{LM}) = H^*(LM; \mathbb{Q})$, and by (6.4),

$$\begin{aligned} Dlcop((1 \otimes su_2) \otimes (1 \otimes 1)) &= H(\bar{\varepsilon} \otimes 1)H(\Delta^! \otimes 1)H(\bar{\varepsilon} \otimes 1)^{-1}H(\mu \otimes_{\mu'} \bar{\zeta})((1 \otimes su_2) \otimes (1 \otimes 1)) \\ &= H(\bar{\varepsilon} \otimes 1)H(\Delta^! \otimes 1)H(\bar{\varepsilon} \otimes 1)^{-1}(1 \otimes (1 \otimes 1 \otimes su_2) \otimes (1 \otimes 1 \otimes 1)) \\ &= H(\bar{\varepsilon} \otimes 1)H(\Delta^! \otimes 1)((-1 \otimes 1 \otimes su_2) \otimes (1 \otimes 1 \otimes 1) \otimes (1 \otimes 1 \otimes 1) \\ &\quad + (1 \otimes 1 \otimes 1) \otimes (1 \otimes 1 \otimes su_2) \otimes (1 \otimes 1 \otimes 1)) \\ &= H(\bar{\varepsilon} \otimes 1)((g \otimes 1) \otimes (1 \otimes 1 \otimes 1)) \\ &= u_2^2 \otimes 1 \neq 0. \end{aligned}$$

By (6.1), $Dlp = H(\Delta^! \otimes 1 \otimes 1)H(\bar{\varepsilon} \otimes 1)^{-1}H((\mu \otimes 1) \otimes_{\mu} (\mu \otimes 1))H(\bar{\varepsilon} \otimes 1)^{-1}$. For the non-zero element $1 \otimes sx_2su_2sw_5$ in $H^*(\mathcal{M}_{LM}) = H^*(LM; \mathbb{Q})$,

$$\begin{aligned} Dlp(1 \otimes sx_2su_2sw_5) &= H(\Delta^! \otimes 1 \otimes 1)H(\bar{\varepsilon} \otimes 1)^{-1}H((\mu \otimes 1) \otimes_{\mu} (\mu \otimes 1)) \\ &\quad \left(1_{\mathcal{M}_{MI}} \otimes (1 \otimes 1 \otimes sx_2su_2sw_5) - (1 \otimes 1 \otimes sx_2) \otimes (1 \otimes 1 \otimes su_2sw_5) \right. \\ &\quad + (1 \otimes 1 \otimes su_2) \otimes (1 \otimes 1 \otimes sx_2sw_5) - (1 \otimes 1 \otimes sw_5) \otimes (1 \otimes 1 \otimes sx_2su_2) \\ &\quad + (1 \otimes 1 \otimes sx_2su_2) \otimes (1 \otimes 1 \otimes sw_5) - (1 \otimes 1 \otimes su_2sw_5) \otimes (1 \otimes 1 \otimes sx_2) \\ &\quad \left. + (1 \otimes 1 \otimes sx_2sw_5) \otimes (1 \otimes 1 \otimes su_2) - (1 \otimes 1 \otimes sx_2su_2sw_5) \otimes 1_{\mathcal{M}_{MI}} \right) \\ &= H(\Delta^! \otimes 1 \otimes 1) \\ &\quad \left(1_{\mathcal{M}_{MI}} \otimes (1 \otimes 1) \otimes (1 \otimes sx_2su_2sw_5) - 1_{\mathcal{M}_{MI}} \otimes (1 \otimes sx_2) \otimes (1 \otimes su_2sw_5) \right. \\ &\quad + 1_{\mathcal{M}_{MI}} \otimes (1 \otimes su_2) \otimes (1 \otimes sx_2sw_5) - 1_{\mathcal{M}_{MI}} \otimes (1 \otimes sw_5) \otimes (1 \otimes sx_2su_2) \\ &\quad + 1_{\mathcal{M}_{MI}} \otimes (1 \otimes sx_2su_2) \otimes (1 \otimes sw_5) - 1_{\mathcal{M}_{MI}} \otimes (1 \otimes su_2sw_5) \otimes (1 \otimes sx_2) \\ &\quad + 1_{\mathcal{M}_{MI}} \otimes (1 \otimes sx_2sw_5) \otimes (1 \otimes su_2) - 1_{\mathcal{M}_{MI}} \otimes (1 \otimes sx_2su_2sw_5) \otimes (1 \otimes 1) \\ &\quad + 3((u_2 \otimes 1 + 1 \otimes u_2) \otimes su_2) \otimes (1 \otimes su_2) \otimes (1 \otimes sx_2su_2) \\ &\quad + ((u_2 \otimes 1 + 1 \otimes u_2) \otimes sx_2) \otimes (1 \otimes su_2) \otimes (1 \otimes sx_2su_2) \\ &\quad + ((x_2 \otimes 1 + 1 \otimes x_2) \otimes su_2) \otimes (1 \otimes su_2) \otimes (1 \otimes sx_2su_2) \\ &\quad + ((u_2 \otimes 1 + 1 \otimes u_2) \otimes su_2) \otimes (1 \otimes sx_2) \otimes (1 \otimes sx_2su_2) \\ &\quad + 3((u_2 \otimes 1 + 1 \otimes u_2) \otimes su_2) \otimes (1 \otimes sx_2su_2) \otimes (1 \otimes su_2) \\ &\quad + ((u_2 \otimes 1 + 1 \otimes u_2) \otimes sx_2) \otimes (1 \otimes sx_2su_2) \otimes (1 \otimes su_2) \\ &\quad + ((x_2 \otimes 1 + 1 \otimes x_2) \otimes su_2) \otimes (1 \otimes sx_2su_2) \otimes (1 \otimes su_2) \\ &\quad \left. + ((u_2 \otimes 1 + 1 \otimes u_2) \otimes su_2) \otimes (1 \otimes sx_2su_2) \otimes (1 \otimes sx_2) \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{3}(x_2u_2 \otimes su_2) \otimes (u_2 \otimes sx_2su_2) + \frac{2}{3}(u_2 \otimes su_2) \otimes (x_2u_2 \otimes sx_2su_2) \\
&\quad + \frac{1}{3}(x_2u_2^2 \otimes su_2) \otimes (1 \otimes sx_2su_2) + \frac{1}{3}(1 \otimes su_2) \otimes (x_2u_2^2 \otimes sx_2su_2) \\
&\quad - \frac{1}{3}(u_2^3 \otimes sx_2) \otimes (1 \otimes sx_2su_2) - \frac{2}{3}(u_2^2 \otimes sx_2) \otimes (u_2 \otimes sx_2su_2) \\
&\quad - \frac{2}{3}(u_2 \otimes sx_2) \otimes (u_2^2 \otimes sx_2su_2) - \frac{1}{3}(1 \otimes sx_2) \otimes (u_2^3 \otimes sx_2su_2) \\
&\quad + \frac{2}{3}(x_2u_2 \otimes sx_2su_2) \otimes (u_2 \otimes su_2) + \frac{2}{3}(u_2 \otimes sx_2su_2) \otimes (x_2u_2 \otimes su_2) \\
&\quad + \frac{1}{3}(x_2u_2^2 \otimes sx_2su_2) \otimes (1 \otimes su_2) + \frac{1}{3}(1 \otimes sx_2su_2) \otimes (x_2u_2^2 \otimes su_2) \\
&\quad - \frac{1}{3}(u_2^3 \otimes sx_2su_2) \otimes (1 \otimes sx_2) - \frac{2}{3}(u_2^2 \otimes sx_2su_2) \otimes (u_2 \otimes sx_2) \\
&\quad - \frac{2}{3}(u_2 \otimes sx_2su_2) \otimes (u_2^2 \otimes sx_2) - \frac{1}{3}(1 \otimes sx_2su_2) \otimes (u_2^3 \otimes sx_2) \neq 0.
\end{aligned}$$

Here, $1_{\mathcal{M}_{M^I}} := 1 \otimes 1 \otimes 1$ in \mathcal{M}_{M^I} . Therefore, the dual loop product is non-trivial.

The same calculation described above shows that

$$Dl\text{cop}(Dl\text{cop} \otimes 1)((1 \otimes su_2) \otimes (1 \otimes 1) \otimes (1 \otimes su_2)) = u_2^4 \otimes 1 \neq 0.$$

Therefore, it is an example which $(L\text{cop} \otimes 1)L\text{cop}$ is non-trivial.

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REFERENCES

- [1] M. Chas and D. Sullivan, String topology, preprint (1999), arXiv:math.GT/9911159.
- [2] D. Chataur and L. Menichi, String topology of classifying spaces, preprint (2008), arXiv:math.AT/08010174.
- [3] D. Chataur and J.-C. Thomas, Frobenius rational loop algebra. Manuscripta Math. 122 (2007), no. 3, 305-319.
- [4] R. L. Cohen, V. Godin, A polarized view of string topology, Topology, geometry and quantum field theory, 127-154, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.
- [5] R. L. Cohen, J. D. S. Jones, J. Yan, The loop homology algebra of spheres and projective spaces. Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), 77-92, Progr. Math., 215, Birkhauser, Basel, 2004.
- [6] Y. Félix, S. Halperin, J. -C. Thomas, Gorenstein spaces. Adv. in Math. 71 (1988), no. 1, 92-112.
- [7] Y. Félix, S. Halperin, J. -C. Thomas, Rational Homotopy Theory, Graduate Texts in Mathematics, 205. Springer-Verlag.
- [8] Y. Félix, J. Oprea, D. Tanre, Algebraic models in geometry. Oxford Graduate Texts in Mathematics, 17. Oxford University Press, Oxford, 2008.
- [9] Y. Félix and J. -C. Thomas, String topology on Gorenstein spaces, Math. Ann., 345 (2009), no. 2, 417-452.
- [10] Y. Félix, J. -C. Thomas, M. Vigue-Poirriér, Rational string topology. J. Eur. Math. Soc., 9 (2007), no. 1, 123-156.
- [11] K. Kuribayashi, L. Menichi, Loop products on Noetherian H-spaces, in preparation.
- [12] K. Kuribayashi, L. Menichi and T. Naito, Derived string topology and the Eilenberg-Moore spectral sequence, preprint (2012), arXiv:math.AT/12116833.
- [13] J. McCleary, A user's guide to spectral sequences. Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001.
- [14] A. Murillo, The virtual Spivak fiber, duality on fibrations and Gorenstein spaces, Trans. Amer. Math. Soc. 359 (2007), no. 8, 3577-3587.
- [15] L. Smith, Homological algebra and the Eilenberg-Moore spectral sequence. Trans. Amer. Math. Soc. 129 (1967) 58-93.

- [16] H. Tamanoi, Loop coproducts in string topology and triviality of higher genus TQFT operations. J. Pure Appl. Algebra 214 (2010), no. 5, 605-615.

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